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On the Theory of Visco-Elasticity: A Thermodynamical Treatment of Visco-Elasticity, and Some Problems of the Vibrations of Visco-Elastic Solids

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X. *On the Theory of Visco-Elasticity: a Thermodynamical Treatment of Visco-Elasticity, and Some Problems of the Vibrations of Visco-elastic Solids.*

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(*Communicated by* E. A. MILNE, *F.R.S.*)

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The problem of the damping of vibrations in a solid which is imperfectly elastic was introduced to me by Professor E. A. MILNE. Sir CHARLES SHERRINGTON had asked Professor MILNE for his assistance in the analysis of an experiment which Dr. J. C. ECCLES was performing at the Oxford Physiological Laboratories, on the damping of forced longitudinal vibrations in a muscle. It was proposed, in the first place, to assume that in this experiment muscle displays dynamical properties of the same character as a non-living solid—a similarity between the dynamical behaviour of muscle and that of rubber and glass had already been noticed. According to this view the damping of vibrations in the muscle would be due to the "imperfectness of elasticity" of the muscle; and in order to work out the analysis of the experiment, it is necessary to know the expression for the tension at any point of the muscle at any instant during its *straining*. An examination of the scientific literature dealing with the imperfectness of elasticity of solids did not indicate an immediate solution of the problem.

The present paper contains a theoretical investigation of the dynamical behaviour of imperfectly elastic solids, which I have attempted at Professor MILNE's suggestion. Dr. ECCLES has performed the experiments which originally suggested this investigation, using rubber instead of muscle. The results of these experiments, together with the analysis of the experiment, are contained in a separate and joint paper.

I wish to acknowledge my deep indebtedness to Professor MILNE for his continual help and criticism, and for many valuable suggestions. My thanks are also due to Professor LOVE who drew my attention to certain points needing modification in the dissertation, of which the subject-matter of this paper originally formed part.

In the *introduction* a brief account is given of work on imperfectness of elasticity since 1834, and this work is discussed with a view to the development of a general theory of the dynamical behaviour of imperfectly elastic solids. § 1 contains a thermodynamical development of a general theory of imperfect elasticity, of the type which we call "visco-elasticity." In § 2 are some general theorems on the damped vibrations of visco-elastic solids. The general results of § 2 are illustrated in § 3 by some particular examples; and some illustrative examples of forced motion are also analysed. The examples chosen concern the vibrations of thin cylinders, which is a subject of practical importance, as a wire or thin cylinder is usually found to be the most convenient form of a solid for experimental investigation in this field.

Introduction.

Since 1834, when Vicat* made some observations on the extension of a loaded wire, it has been realised that the mathematical theory of elasticity, founded on the basis of the generalised Hooke's law, is incapable of predicting completely the elastic behaviour of solids. The observed deviations from the behaviour predicted by Hooke's law have formed a subject of frequent investigation, and are of importance in such widely scattered fields as the theory of tides and the engineering problem of the bending of rotating shafts. Existing investigations provide either a phenomenological treatment or a structure-theory treatment.

Recently Prandtl† has initiated a theory of elastic afterworking and hysteresis in crystals, attacking the problem from the side of molecular statistics. But such a development is of far greater complexity and difficulty in the case of the so-called isotropic solids (in which these phenomena are most pronounced) on account of their complex and haphazard structure. We here confine our attention to those investigations depending on a phenomenological treatment. Some of the older structure theories properly fall into this class. The contemporary knowledge of atomic physics was inadequate to supply the details of the molecular structure; and the work of Meyer‡ and of Brillouin,§ for example, seems to be an induction of the molecular structure from the elastic properties of solids, rather than a deduction of the elastic properties of solids from a hypothetical molecular structure, even though the work appeared in the latter form.

The importance of Vicat's paper is that it suggests the idea of elastic limits. He found that a wire, held stretched under a tension equal to one quarter of the breaking stress, remained of constant length throughout the time of his experiments (nearly three years), though similar wires showed a gradual increase of length when under a tension equal to one-third of the breaking stress—2·75 cm. per metre in 33 months. The so-called "limit of perfect elasticity" is determined by the greatest limiting strain such that the solid acquires no residual permanent deformation (set) through being strained, provided that the strain at every point of the solid never exceeds the limiting

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* Vicat. "Annales des ponts et chaussées," 1834, 1er semestre, p. 40.

† L. Prandtl. 'Z. angew. Math. Mech.,' Vol. 8, p. 85 (1928).

‡ 'Crelle's J.,' vol. 78, p. 130 (1874); vol. 80, p. 315 (1875).

§ 'Ann. Chim. (Phys.),' vol. 13, p. 377; vol. 14, p. 311; vol. 15, p. 447 (1898).

strain. When a solid is strained beyond the "limit of perfect elasticity," it is said to be plastic. In this state it is found that the strain gradually increases under a fixed stress, and that a permanent set remains after release from a state of stress.

Now the essential feature of the classical theory of elasticity is that the elastic solid described by the theory is a conservative mechanical system. It follows that the stress components in such a solid are determined by the instantaneous values of the generalized co-ordinates which define the instantaneous strained configuration, *e.g.*, in the case of isothermal straining, by the strain components alone. We here define the ideal solid characterised by this property as "perfectly elastic." The definition of the so-called "limit of perfect elasticity" implies that, for equilibrium—and *for equilibrium only*—the stress components are determined by the instantaneous strained configuration. In our terminology this limit is thus more accurately described as the "limit of perfect *statical* elasticity," and a solid may display a *dynamical* imperfectness of elasticity when strained within this limit.

The experimentally determined value of the limit of perfect (statical) elasticity depends on the degree of experimental error in its determination. In order to define the limit precisely it is now defined as the greatest limiting strain such that the residual strain (or set) does not exceed a very small but measurable proportion (0·03 per cent.) of the limiting strain.* Without this convention, it is clear that the value of the limit would be reduced by increased experimental accuracy, and it is a matter of fundamental importance whether the true limit has a zero or non-zero value, as determined in the ideal case of perfect experimental accuracy.

We confine our attention to the behaviour of solids within the *practically defined* and *experimentally determined* limit of perfect (statical) elasticity. It is necessary to consider two possibilities. The true limit may have a zero value, in which case there is a true statical imperfectness of elasticity (the effects of plasticity are never absent), and the classical theory of elasticity needs modification even for the case of a strained solid in equilibrium. If the true limit has a non-zero value, then within this limit there is only a dynamical imperfectness of elasticity, and the classical theory of elasticity describes accurately the behaviour of a strained solid in equilibrium, but may need modification to describe accurately the *dynamical* behaviour of a *straining* solid. But, irrespective of whether the true limit has a zero or non-zero value, there would still appear to be a difference between the behaviour of a solid within its practically determined limit of perfect (statical) elasticity and its behaviour outside this limit. For, when a solid is strained only within this limit, any permanent set is too small to measure; but if the solid is strained beyond this limit, the magnitude of the permanent set is limited only by the breaking point. Thus, while to assert that the discrepancy between the observed dynamical behaviour of solids, within the practically defined limit of perfect (statical) elasticity and the dynamical behaviour predicted by HOOKE's law,

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* See, for example, the definition of the "Sichtbare Elastizitätsgrenze" in TH. v. KÁRMÁN'S "Physikalische Grundlagen der Festigkeitslehre," 'Enc. Math. Wiss.,' 4.2.2.5., footnote 23 (γ), p. 704.

is not owing to the same molecular (or microscopic) physical cause as the well-known plastic effects outside this limit would require that the true limit has a non-zero value, it might still be expected that the behaviour of solids within this practically defined limit could be capable of an approximate mathematical description, while their behaviour beyond this limit would not be capable of the *same* mathematical description.

The two principal effects in the observed dynamical behaviour of elastic solids, which are inconsistent with the behaviour predicted by HOOKE's law, are :—

(1) The damping of vibrations in solids.

(2) Hysteresis phenomena. (This term is taken to include all phenomena which indicate a tendency on the part of the solid to delay in taking up the strain system which the instantaneous stress system demands according to HOOKE's law. In this category must be included some of the " Elastiche Nachwirkung " or elastic afterworking effects, though the term is sometimes used in such a wide sense in German literature as to include the phenomenon of permanent deformation.)

The phenomenon of elastic afterworking appears to have been first noticed (in 1835) by WEBER,* who found that when a silk fibre is stretched and then released, it does not immediately return to its unstretched position, but is left with an extension which decreases asymptotically to zero with the increase of time. R. KOHLRAUSCH† performed similar experiments, but it was the work of F. KOHLRAUSCH‡ which was the first of a series of more exact investigations to determine empirical laws concerning the phenomena first noticed by VICAT and WEBER. F. KOHLRAUSCH performed three series of experiments on twisted glass wires. The results of the second series showed that the " torsional afterworking " of the wire after release from a state of torsion could be expressed by a formula $x = c/t^{\alpha}$, where α is a constant known as the " coefficient of elastic afterworking," and the time t is measured from the instant of release. The constant c depends on the magnitude and duration of the twist preceding release. This result does not appear to be consistent with the result of the third series of experiments according to which the couple necessary to keep the wire in an untwisted state after release from a state of torsion is of the form $d = d_1 + ce^{-at^m}$. The term d_1 indicates a permanent set. Again the formula $x = c/t^{\alpha}$ for the afterworking leads to the obviously absurd result that $x \rightarrow \infty$ when $t \rightarrow 0$. NEESEN's§ expression of the afterworking by a formula $x = ce^{-\beta t} + c_1 e^{-\beta_1 t}$ seems preferable. This more complicated formula was criticised by F. KOHLRAUSCH|| on the grounds that it fitted the results no better than his own simpler formula $x = c/t^{\alpha}$, but this can only be the case if t is not measured small enough to expose the inadequacy of the simpler formula.

The field of investigation of imperfect elasticity was widened, when in 1865 Lord

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* 'Pogg. Ann.,' vol. 34, p. 247 (1835); vol. 54, p. 1 (1841).

† 'Pogg. Ann.,' vol. 72, p. 393 (1847).

‡ 'Pogg. Ann.,' vol. 119, p. 337 (1863); vol. 128, pp. 1, 207 and 399 (1866); vol. 158, p. 337 (1876).

§ 'Pogg. Ann.,' vol. 153, p. 498 (1874).

|| 'Pogg. Ann.,' vol. 155, p. 579 (1875).

KELVIN* published a paper containing an investigation of the damping of the vibrations of torsional pendulums. As a result of certain arguments and preliminary experiments, he came to the conclusion that the damping of vibrations was due almost entirely to dissipation of energy in the wire itself, and that dissipation of energy owing to air resistance, thermo-elastic causes, and lack of rigidity at the point of fixation could only account for a very small proportion of the observed damping. He examined the experimental results in the light of a hypothesis which he suggested. This was, that when a solid is *straining* viscous forces are called into play, which are strictly analogous to the viscous stresses in a liquid. This idea of the extension of the theory of viscosity from the liquid state to the solid state had already been suggested, rather indirectly, in 1845 by STOKES,† who remarked that "there appears to be no line of demarcation between a solid and a viscous fluid." KELVIN defined solid viscosity as molecular friction, and pointed out that the word viscous had sometimes been used to describe properties which might more accurately have been termed plastic. This wide use of the term viscous has unfortunately continued,‡ but some restriction as to its use is necessary to avoid confusion. It is convenient here to restrict the use of the word "viscous" to solids in which there is an additional stress owing to molar motion, of the same form as for a fluid. As a result of his experiments, KELVIN found that :—

(1) In any series of experiments, with the same torsional pendulum, for small amplitudes of vibration, the logarithmic decrement of the amplitude per vibration was constant.

(2) In similar experiments, but with different vibrators, the logarithmic decrement was not quite that expected according to STOKES'S viscosity law, which predicts that the logarithmic decrement per vibration would be inversely proportional to the period of vibration. The logarithmic decrement was found to be greater for the longer periods than according to this law.

KELVIN also found certain fatigue effects in wires that were kept vibrating continuously, but some doubt was thrown on the value of his experimental results by THOMPSON.§ Maintaining the vibrations by an electrical device when investigating fatigue effects, he found that neither the logarithmic decrement nor period of vibration depended on the previous elastic history of the wire; and suggested that the wires must have been excessively twisted in KELVIN'S experiments—a reasonable suggestion in view of the fact that some of KELVIN'S experiments were performed by students.

On the assumption that the action between two particles is only transmitted at a

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* Sir W. THOMSON, 'Proc. Roy. Soc.,' vol. 14, p. 289 (1865); reprinted in his collected works, "Math. and Phys. Papers," vol. 3, p. 27 (Cambridge 1890).

† 'Trans. Camb. Phil. Soc.,' vol. 8, p. 287 (1845). Reprinted in his "Scientific Papers," vol. 1, p. 75.

‡ For example, the use of the word "elastico-viscous" by J. G. BUTCHER ('Proc. Math. Soc. Lond.,' vol. 18, p. 103 (1877)) and others; *cf.* also "Viscous Flow of Metals," ANDRADE, 'Proc. Roy. Soc.,' A, vol. 84, p. 1 (1910).

§ 'Phys. Rev.,' vol. 8, p. 141 (1899).

finite rate, MEYER (*loc. cit.*) worked out the resulting expressions for the stress components in a solid, using the original Cauchy structure as a basis of his calculations; and found that the stress components were expressed linearly in terms of the strain components and *rate of strain* components. MEYER* later attempted to explain the phenomenon of elastic afterworking according to this set of stress-strain relations.

BOLTZMANN† criticised MEYER'S explanation of elastic afterworking, and suggested a modification of the generalised HOOKE'S law by which he attempted to explain both the damping vibrations in solids and the elastic afterworking phenomenon. BOLTZMANN'S modifications involve an extension of the "relaxation hypothesis," due to MAXWELL,‡ and which appeared in the latter's paper "On the Dynamical Theory of Gases." MAXWELL'S suggestion was that, while the HOOKE'S law stress-strain relation in its simplest form is $F = ES$ (F is the stress, S the strain, and E an elastic modulus), the actual behaviour of a solid is more closely predicted by the relation

$$\frac{dF}{dt} = E\frac{dS}{dt} - \frac{F}{T},$$

where T is constant for any particular solid, and of the dimensions of a time. Thus, under a given stress the strain increases indefinitely; and the stress required to maintain a given strain decreases exponentially to zero. The relaxation hypothesis is clearly an attempt to ascribe all imperfectness of elasticity as due to the effect of plasticity, and requires that the true limit of perfect (statical) elasticity should have a zero value. The hypothesis cannot be dismissed on these grounds, but the results that solids flow indefinitely under a given stress and that a 100 per cent. permanent set is possible would seem to make the hypothesis inapplicable to metals and all materials used in building. It is difficult to believe that the hypothesis is applicable even to glass in its *durovitreous* state, since a small flow under its own weight (as the hypothesis requires) would be immediately discernible in optical instruments, while, on the other hand, the elastic afterworking effects in glass are most pronounced.

Like MAXWELL, BOLTZMANN modified the HOOKE'S law stress-strain relation by the introduction of "relaxation terms," but he used a different expression for these terms. The couple L, to be applied to the free end of a wire to twist that end through an angle θ, is given by

$$L = K\,\theta$$

according to HOOKE'S law (where K is a constant). According to BOLTZMANN'S hypothesis, if the wire had been twisted through an angle $\theta(\tau)$ at time τ for a period of time $d\tau$, there is a relaxation so that the couple $L(t)$, required to maintain a twist $\theta(t)$ at time t, is

$$L(t) = K\,\theta(t) - \psi(t-\tau)\,\theta(\tau)\,d\tau,$$

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* 'Pogg. Ann.,' vol. 151, p. 108 (1874).

† 'Pogg. Ann. Ergänzungsbd.,' vol. 7, p. 624 (1876).

‡ 'Phil. Trans.,' vol. 157, p. 49 (1867). Reprinted in his "Scientific Papers," vol. 2, p. 26.

where $\psi(t-\tau)$ is an undetermined function of the interval of time $(t-\tau)$. He further assumed that these relaxation effects are superposed, and putting $t-\tau=\omega$, arrived at the following modified form of the stress-strain relations in an isotropic solid :—*

$$p_{xx}(t)=\lambda\,\Delta(t)+2\mu e_{xx}(t)-\int_0^\infty d\omega\,[\phi(\omega)\,\Delta(t-\omega)+2\psi(\omega)\,e_{xx}(t-\omega)],\ \text{etc.};$$

$$p_{yz}(t)=2\mu\,e_{yz}(t)-2\int_0^\infty d\omega\,\psi(\omega)\,e_{yz}(t-\omega),\ \text{etc.}$$

The value of the stress components thus depends only on the instantaneous strain and the past strain history. The two functions $\phi(\omega)$ and $\psi(\omega)$ are as yet undetermined. BOLTZMANN first attempted to determine the function $\psi(\omega)$ by experiments on torsional afterworking. His experiments suggested a form A/ω for $\psi(\omega)$; and he took $\psi(\omega)=f(\omega)/\omega$, where $f(\omega)$ is constant for moderate ω, but has to be modified for small and large values of ω in order to ensure the convergence of the integral $\int_0^\infty \psi(\omega)\,d\omega$. The value of $f(\omega)$ for large values of ω was determined by the assumption $\mu=\int_0^\infty \psi(\omega).d\omega$. This leads to the same result as MAXWELL's relaxation hypothesis, namely, that the couple necessary to maintain a fixed twist decreases exponentially to zero with the increase of time, thus giving a 100 per cent. permanent set. And the assumption is therefore in direct contradiction to the fact that $\int_0^\infty \psi(\omega).d\omega$ must not have a value exceeding 0·03 per cent. of the value of μ, when the solid is strained only within the practically defined limit of perfect (statical) elasticity. From this determination of $\psi(\omega)$, and assuming $\phi(\omega)=\psi(\omega)$, BOLTZMANN calculated a value of the logarithmic decrement of a torsional pendulum. The observed logarithmic decrement was about 50 per cent. greater than the calculated, a difference too large to be explained by air resistance; and the calculated value would already appear to be too large on account of the assumption $\mu=\int_0^\infty \psi(\omega)\,d\omega$. BOLTZMANN's theory predicted that the logarithmic decrement of the amplitude of vibration of a torsional pendulum would be constant and independent of the period of vibration, a result which does not appear to be true.† The fact that the principle of superposition is not verified by experiments on elastic afterworking‡ is also a serious criticism of the theory. The differential equations of motion of the theory of elasticity become integro-differential equations with BOLTZMANN's modifications of HOOKE's law, and VOLTERRA§ has developed the resulting mathematical theory in several papers.

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* This notation for stress and strain is defined in § 3·1.

† KLEMENCIC, 'Wien Ber.,' vol. 81 (2), p. 791 (1880).

‡ HOPKINSON, 'Proc. Roy. Soc.,' vol. 28, p. 148 (1878).

§ VOLTERRA, "Leçons sur les équations intégrales et les équations intégro-differentielles," Paris, 1913. For references to individual papers, see LOVE's 'Elasticity,' p. 121 (4th Edition, Camb., 1927).

MAXWELL'S relaxation hypothesis was derived (in a more general form) from a structure theory by BUTCHER.* The details of the suggested structure were due to MAXWELL. It was assumed that molecules in a solid group together in two ways, the groups of the first type being stable until a certain limiting strain is applied, while groups of the second type are continually splitting up. A viscous fluid is composed of molecular groups of the second type only, and a solid composed of molecular groups of both types was called "*elastico-viscous*." It is clear from the assumed molecular structure as well as from the resulting stress-strain relations that an *elastico-viscous* solid is only a particular type of a *plastic* solid. The attempt to explain all imperfectness of elasticity of solids as due to plasticity *of the elastico-viscous type* is open to the same criticisms as was the more elementary relaxation hypothesis due to MAXWELL. The same remarks apply to BRILLOUIN'S† "Theory of the Double Medium," which is the molar analogue of the elastico-viscous structure theory.

In 1892, VOIGT‡ published a long paper on the "Innere Reibung" of metals. He supposed that the stress components in a strained and *straining* solid could be expressed as the sum of two sets of terms, the first set being the ordinary HOOKE'S law elastic stress, the second set representing a generalised viscous stress. Thus the stress components are expressed as the sum of terms linear in the strain components and the *rate of strain* components, according to§

$$p_{\alpha\beta} = c_{\alpha\beta mn} e_{mn} + a_{\alpha\beta mn} \frac{\partial}{\partial t} e_{mn}.$$

We call these relations the "*visco-elastic*" stress-strain relations.|| VOIGT said that there were 36 distinct "*viscous coefficients*," $a_{\alpha\beta mn}$, though we shall see (§ 1·6) that an additional symmetry relation reduces this number to 21 in the general case of an æolotropic solid. In the case of an isotropic visco-elastic solid, the stress-strain relations reduce to :—

$$p_{xx} = \lambda \Delta + 2\mu e_{xx} + \lambda' \frac{\partial \Delta}{\partial t} + 2\mu' \frac{\partial e_{xx}}{\partial t}, \text{ etc.};$$

$$p_{yz} = 2\mu e_{yz} + 2\mu' \frac{\partial e_{yz}}{\partial t}, \text{ etc.}$$

VOIGT analysed the problem of the bending and torsional vibrations of thin isotropic visco-elastic rods, fixed at one end and having an inertial disc attached at the "free end." The analysis was based on the assumption that the "viscous terms" in the

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* 'Proc. Math. Soc. Lond.,' vol. 8, p. 103 (1877).

† 'Ann. Chim. (Phys.),' vol. 13, p. 377 ; vol. 14, p. 311 ; vol. 15, p. 447 (1898).

‡ 'Ann. Phys.,' vol. 47, p. 671 (1892).

§ This tensor notation is defined in § 1·1 and § 1·2.

|| Terminology is at present confused. SEZAWA and HOSALI have used "*visco-elastic*" in this sense, but JEFFREYS uses the word "*firmo-viscous*." The term "visco-elastic" seems to describe the relations as accurately and as fully as possible. Care must be taken to avoid confusion with the rather unfortunate term "*elastico-viscous*," which has an entirely different meaning.

expression for the stress components are small compared with the "elastic terms." It was shown that the logarithmic decrement of the amplitude per vibration should be inversely proportional to the period of vibration (for either bending or torsional vibrations), in contrast to the constancy of the logarithmic decrement which BOLTZMANN's hypothesis predicted. This provided an interesting criterion by which to test the applicability of the two theories. After making corrections for air resistance, VOIGT came to the conclusion that the damping of both bending and torsional vibrations could be explained on the visco-elastic hypothesis for copper and nickel, as could the damping of the bending vibrations for brass and bronze. Cadmium appeared to behave according to BOLTZMANN's hypothesis, while the results for the cast metals fit in with neither. From the results of his experiments, VOIGT was able to calculate the values of both λ' and μ' the viscous coefficients corresponding to the LAMÉ elastic constants λ and μ. For nickel he found that

$$\lambda' = 448 \times 10^6, \quad \mu' = 12\cdot 5 \times 10^6;$$

while, from his data, λ and μ can be roughly calculated as

$$\lambda = 8 \times 10^{11}, \quad \mu = 6 \times 10^{11}.$$

It is important to notice that neither of the relations

$$\lambda' + \tfrac{2}{3}\mu' = 0 \quad \text{or} \quad \lambda'/\lambda = \mu'/\mu$$

is satisfied even approximately by these experimental results.* Both of these relations have been assumed to be true by recent investigators.†

By comparison of an elastic medium with a monocyclic system, REISSNER‡ obtained all the well-known results for an elastic solid in a very concise way. He proposed to account for the imperfectness of elasticity of solids by the introduction of terms, depending on the rate of strain, into the expression for the "energy" of the solid. Owing to the abstract nature of this modification, it is difficult to see whether or not the resulting theory is likely to explain the observed dynamical behaviour of solids, and what, if any, the limits of the applicability of the theory would be.

Since 1900, investigations of imperfectness of elasticity have usually dealt with particular problems, often of industrial importance. Little new work of general theoretical interest has been attempted, and the particular problems have generally

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* The relation $\lambda' + \frac{2}{3}\mu' = 0$ means that there is no dilatational viscosity and would reduce the additional viscous stress to exactly the same form as STOKES's viscous stresses in a liquid. STOKES arrived at this result by a symmetry argument. There does not seem to be any reason for adopting this result for a solid, unless some molecular cause for the viscous terms is pre-supposed.

† See footnotes ** p. 348; §, ||, ¶, ** p. 349.

‡ 'Ann. Phys.,' vol. 9, p. 44 (1902).

been examined in the light of VOIGT's visco-elastic hypothesis or MAXWELL's relaxation (elastico-viscous) hypothesis.

Two papers, published in 1921, contain accounts of detailed investigations of the damping of vibrations of thin rods, the results being examined in the light of the visco-elastic hypothesis. In the first, HONDA and KONNO* found that for the various metals which they examined (1) the "coefficient of normal viscosity"† is of the same order of magnitude (10^8) as the "coefficient of tangential viscosity,"‡ (2) the coefficients of viscosity of a metal having a low melting point are large.

The second paper, by IOKIBE and SAKAI,§ gives the results of an experimental investigation of the variation with temperature of the elastic and viscous coefficients, and contains a full account of previous work on this subject. The experiments were made on the damping of the vibrations of torsional pendulums. The slight discrepancies between the observed damping and that predicted by the visco-elastic hypothesis were supposed to be due to dissipation of energy at the point of fixation, and to the fact that the amplitude of the twist was excessively great. The "real" logarithmic decrement was obtained by a method of interpolation. The tangential viscosity of a solid was found to increase with increase of temperature, in general; but in the case of iron (which has a high melting point) it was found to decrease to a minimum before starting to increase.

HETTWER‖ has calculated the coefficient of tangential viscosity from observations on slow forced torsional vibrations of wires, and obtained values of the order of 10^{15} for various metals, in contrast to the values of the order of 10^8 obtained by investigators from experiments on the damping of vibrations of torsional pendulums.

The analysis of the problem of the longitudinal vibrations of a thin rod has been worked out by CADY¶ and QUIMBY,** but both solutions are open to criticism. CADY uses the equation of motion of sound waves in a viscous medium,†† and assumes that it is the equation of motion of the longitudinal vibrations of a visco-elastic rod, on account of the well-known mathematical analogy between the two problems in the case of perfect elasticity. QUIMBY assumes in his analysis that the ratio of the lateral to longitudinal strain is $-\sigma$ (POISSON's ratio). As we shall see in § 3·4, this is false unless there is a relation $\lambda'/\lambda = \mu'/\mu$ between the elastic and viscous coefficients; and QUIMBY had already assumed that there is no dilatational viscosity, *i.e.* $\lambda' + \frac{2}{3}\mu' = 0$. From observations of the damping of the vibrations (which were exceedingly rapid)

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* 'Phil. Mag.,' vol. 42, p. 115 (1921).

† The coefficient of normal viscosity is analogous to YOUNG's modulus. HONDA and KONNO were here using the results of VOIGT's approximations. This point is dealt with in § 3·4.

‡ The coefficient of tangential viscosity is μ'.

§ 'Phil. Mag.,' vol. 42, p. 397 (1921).

‖ 'Wien Ber.,' vol. 134, p. 51 (1925).

¶ 'Phys. Rev.,' vol. 19, p. 1 (1919).

** 'Phys. Rev.,' vol. 25, p. 559 (1925).

†† See RAYLEIGH's "Theory of Sound," vol. 2, p. 283.

and using this analysis, he obtained a value of the order of 10^3 for the tangential coefficient of viscosity of glass and copper.

Wartenberg[*] has observed that a metal wire composed of large crystals does not exhibit observable elastic afterworking or hysteresis, as he also found to be the case for any wire at the temperature of liquid air. Okubo[†] has worked out a theory of elastic afterworking on the assumption that this phenomenon is due to slipping at crystal interfaces, and for some simple experiments, the observed results agreed with those predicted by his theory. This attempt to ascribe imperfectness of elasticity as being caused by action at crystal interfaces is in agreement with the known fact that individual crystals exhibit very little afterworking or damping of vibrations.

One of the most interesting applications of the theory of imperfectness of elasticity of solids is to geophysical problems. Sir G. H. Darwin[‡] worked out the theory of tides of an elastico-viscous spheroid in connection with his "Theory of Tidal Friction." Concerning this treatment of the earth as elastico-viscous he says that the elastico-viscous stress-strain relation "has not . . . . any experimental justification; its adoption was rather due to mathematical necessities than any other reason."

Jeffreys[§] has considered the possibility of the representation of the imperfect elasticity of the earth by the elastico-viscous stress-strain relation (Maxwell's relaxation hypothesis) or by what he calls the "firmo-viscous" law (which is Voigt's visco-elastic hypothesis with the assumption that there is no dilatational viscosity), or by a combination of the two. In a recent paper Jeffreys[||] has applied the visco-elastic hypothesis to the problem of the propagation of seismic waves; and from the time of growth of distortional waves, he deduces a rough value of 0·004 seconds for the ratio μ'/μ for the earth.

Hosali[¶] and Sezawa[**] have also investigated the problem of the damping of waves in a visco-elastic solid. Hosali used the method of solution by series, whereas Jeffreys used the operational calculus; both assumed that there was no dilatational viscosity. Hosali solved the problem of the free vibrations of a visco-elastic sphere; and showed that the vibration can be represented by the superposition of vibrations, some of which are exponentially damped harmonic oscillations and other exponentially damped aperiodic motions. Sezawa's paper, which is purely mathematical, is very comprehensive. It includes the solution of the problem of the propagation of dilational and surface waves from a given initial disturbance, and thus deals with the difficulties that arise from attempting to satisfy initial conditions for the first time. The method

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[*] 'Verh. deuts. phys. Ges.,' p. 113 (1918).

[†] 'Sci. Rep. Tohoku Univ.,' vol. 11, p. 173 (1922).

[‡] 'Phil. Trans.,' vol. 170, p. 1 (1879).

[§] 'Mon. Not. R. Astr. Soc.,' vol. 75, p. 648 (1915); vol. 77, p. 449 (1917); *cf.* also "The Earth," Camb., p. 263, 1929.

[||] 'Geophys. Suppl. R. Astr. Soc.,' vol. 2, p. 318 (1931).

[¶] 'Proc. Roy. Soc.,' A, vol. 104, p. 271 (1923).

[**] 'Bull. Earthq. Res. Inst. Tokyo Univ.,' vol. 3, p. 43 (1927).

used is the FOURIER integral method; and it seems that the results are only valid for certain types of initial disturbances, as otherwise the analysis deals with divergent integrals. By taking only an approximate form of the general equations of wave motion, errors are introduced which may be considerable for some types of initial disturbance. The problem of the propagation of surface waves introduces the difficulty of satisfying the surface (boundary) conditions, and in order to satisfy them, SEZAWA is forced to make the assumption that $\lambda'/\lambda = \mu'/\mu$. He showed that, however sharp the initial form of disturbance may be, the pulses in a visco-elastic solid assume gradually flat forms, their apparent wave-length being prolonged (as compared with the corresponding pulses in a perfectly elastic solid). SEZAWA also analysed the problem of the propagation of waves in a beam fixed at one end. Though he does not give the derivation of his equation of motion, it is clear that he has made some approximation such as VOIGT made, or else assumed that $\lambda'/\lambda = \mu'/\mu$ as is implied by QUIMBY's analysis of a similar problem.

Now, any molar (or large scale) theory of the properties of a strained and straining solid can be nothing else but the "*thermodynamics of a continuous medium.*" It must be emphasized that the stress-strain relations are not the basis but the result of such a theory. A stress-strain hypothesis, as it stands, can tell us very little about the properties of the solid it claims to represent, and it is not necessarily consistent with the first and second laws of thermodynamics or with the fact that the solid is to be regarded as continuous. For instance, certain symmetry relations must hold between the elastic coefficients (or constants) occurring in the expression of HOOKE's law; and we shall see in § 1·6 that equivalent symmetry relations must hold between the viscous coefficients in VOIGT's visco-elastic stress-strain relations. Further, the mere statement of a stress-strain hypothesis gives no suggestion as to the limits in which it is valid, whereas a thermodynamical theory can sometimes supply information concerning the character of the limits.

The value of experimental evidence in the development of a thermodynamical theory of a continuous medium applicable to a strained and straining solid is twofold. In the first place the character of the experimental evidence suggests the functional form for the expression for the energy of the solid, and also for the rate of dissipation of energy (if any—there is none according to the classical theory of elasticity). In the second place, quantitative experimental results indicate the approximations that must be made in reducing these unknown functions to tractable form. Thus the fact that an elastic solid strained within the limit of perfect (statical) elasticity always returns to the same equilibrium position when released from a strained position can be shown to justify the expression of the "strain-energy" in terms of the geometrical strain. The quantitative relation expressed by HOOKE's law (even in its simplest form) suggests that, in the expression for the strain-energy, powers of the strain components of order higher than the second can be neglected.

In § 1, a theory of the thermodynamics of a continuous medium is worked out in such a form as to describe the behaviour of a strained and straining solid within the practically defined limit of perfect (statical) elasticity, and, in particular, the vibrations of solids. We now examine the experimental data of the imperfectness of elasticity to obtain the information which is needed in the construction of such a theory.

It seems certain that dissipation of energy occurs inside a solid when it is straining, however small the value of the strain during the straining may be. No other explanation of the damping of the vibrations of a torsional pendulum is tenable. With regard to the quantitative expression of the rate of dissipation of energy, we remark that the visco-elastic hypothesis, due to VOIGT, has met with considerable success in explaining the observed damping of vibrations in individual sets of experiments. The value of the coefficient of tangential viscosity of metals obtained by HETTWER was of the order of 10^{15}; the value obtained by HONDA and KONNO and other investigators of the damped vibrations of torsional pendulums was of the order of 10^8; and the value obtained by QUIMBY was of the order of 10^3. It is significant that the decreasing order of magnitude 10^{15}, 10^8, 10^3 corresponds to the increasing order of the speeds of vibration in the respective sets of experiments; though the value obtained by QUIMBY may be open to criticism, and lack of rigidity of fixation would have the greatest effect on HETTWER's result. A possible inference is that the visco-elastic hypothesis may be approximately accurate for a limited range of rate of change of strain, but is only an approximate form of some more general law.

In view of these experimental results, it seems reasonable to suppose, firstly that the rate of dissipation of energy depends on the rate of straining, and secondly that an expression for the dissipation of energy which leads to the visco-elastic stress-strain relations is approximately correct for some range of rate of straining.

Experimental evidence which is of use in the formulation of the expression for the energy of a strained solid is much less precise. The important question as to whether the true limit of perfect (statical) elasticity has a zero or non-zero value is still unanswered. The most definite knowledge is drawn from the practical definition of this limit. Evidence drawn from experiments on elastic afterworking effects is indirect, and its value is difficult to assess as excessive straining appears to have occurred in some of the experiments.

§ 1.—A THERMODYNAMICAL DEVELOPMENT OF A THEORY OF VISCO-ELASTICITY.

1·1. *Stress: notation.*

A set of rectangular cartesian axes, fixed in space, are taken as axes of reference. The origin is at O, and the directions of the axes are denoted by the numbers 1, 2, 3.

The α-component of internal force across a plane element of area dS is denoted by $p_{\nu\alpha}\,dS$, where ν indicates the direction of the normal to dS. $p_{\nu\alpha}$ is taken as positive when the internal stress is a tension, negative when it is a pressure. The components $p_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ are components of the stress tensor $\mathbf{p}$.

1·2. *Strain and rotation: notation.*

The geometrical configuration of a solid body is determined when the position in space of the particles of which it is composed is known. The standard configuration is taken to be one of equilibrium and uniform temperature θ_0. In this standard configuration the position of a typical particle P is determined by the *vector* OP, denoted by $\mathbf{x}$. Its components along the axes are x_α ($\alpha = 1, 2, 3$). The vector OP at any instant of time t is denoted by $\mathbf{x} + \mathbf{u}$, and its components along the axes by $x_\alpha + u_\alpha$ ($\alpha = 1, 2, 3$). The vector $\mathbf{u}$ is thus the displacement of P at time t.

The position of a neighbouring particle P′ relative to P is measured by the vector $d\mathbf{x}$ in the standard configuration, and by the vector $d\mathbf{x} + d\mathbf{u}$ at time t. Thus $d\mathbf{u}$ measures the displacement of P′ relative to P at time t.

For the component du_α we have

$$du_\alpha = \frac{\partial u_\alpha}{\partial x_\beta} dx_\beta.$$

(A repeated dummy suffix indicates summation over all possible values of that suffix.)

Thus

$$du_\alpha = e_{\alpha\beta}\, dx_\beta + \omega_{\alpha\beta}\, dx_\beta, \quad \dots \dots \dots \quad (1\cdot21)$$

where

$$e_{\alpha\beta} = \tfrac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right), \quad \dots \dots \dots \quad (1\cdot22)$$

and

$$\omega_{\alpha\beta} = -\tfrac{1}{2}\left(\frac{\partial u_\beta}{\partial x_\alpha} - \frac{\partial u_\alpha}{\partial x_\beta}\right)$$

$$= -\mathrm{E}_{\alpha\beta\gamma}\varpi_\gamma \quad \dots \dots \dots \quad (1\cdot23)$$

$\mathrm{E}_{\alpha\beta\gamma}$ is a component of the **E**-tensor, sometimes called the "alternate tensor,"* and

$$\boldsymbol{\varpi} = \tfrac{1}{2}\,\mathrm{curl}\,\mathbf{u}. \quad \dots \dots \dots \quad (1\cdot24)$$

Hence

$$du_\alpha = e_{\alpha\beta}\, dx_\beta - \mathrm{E}_{\alpha\beta\gamma}\varpi_\gamma\, dx_\beta,$$

so that

$$d\mathbf{u} = \mathbf{e}\,.\,d\mathbf{x} - d\mathbf{x} \wedge \boldsymbol{\varpi}$$

$$= \mathbf{e}\,.\,d\mathbf{x} + \boldsymbol{\varpi} \wedge d\mathbf{x}. \quad \dots \dots \dots \quad (1\cdot25)$$

The tensor $\mathbf{e}$ is called the strain tensor, $\boldsymbol{\omega}$ the rotation tensor, and $\boldsymbol{\varpi}$ the rotation vector.

From equation (1·25) we see that the relative displacement of two particles P and P′ at time t, as compared with the standard configuration, is the sum of two terms, $\mathbf{e}\,.\,d\mathbf{x}$ and $\boldsymbol{\varpi} \wedge d\mathbf{x}$. The second term represents a rotation $\boldsymbol{\varpi}$ of the line PP′. The first term is easily seen to represent a change in length of PP′, the change of length varying

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* A component $\mathrm{E}_{\alpha\beta\gamma}$ is zero if any two suffixes are equal and equal to ± 1 if all three are different, the upper or lower sign being taken according as the number of inversions in α, β, γ of the natural order, 1, 2, 3, is even or odd.

with the direction of PP′, and expressible by means of the strain tensor **e**, which thus completely determines the change of shape of the immediate surroundings of P.

1·3. *Application of the Principle of Virtual Work : the energy of a strained solid.*

Now, it is well known that all the properties of the motion of a *mechanical* system can be deduced from the application of the principle of virtual work. In this case a virtual change of configuration of the system is considered subject to the law of conservation of mechanical energy, which is the appropriate special form of the more general first law of thermodynamics.

The theory of the change of configuration of a compressible solid is not simply mechanical but thermodynamical, as a change of strain is, in general, accompanied by a change of temperature. We proceed, therefore, to investigate the properties of strained and straining solids by the complete application of the principle of virtual work, *i.e.*, the consideration of a virtual change of configuration of a solid subject to the first two laws of thermodynamics.

We now consider a solid of any given shape and size, and suppose, for simplicity, that there is no field of force (*i.e.*, there are no body forces or body moments on an element of the solid due to the external field). As there is no field of force, we take the standard configuration to be one of stable equilibrium and uniform temperature, and further suppose that there is no internal stress and that no forces are applied to the surface of the body.*

The energy of a solid is a concept which is a logical consequence of the first law of thermodynamics. The excess of energy of a solid in any configuration over its energy in the standard configuration is equal to the amount of energy (in the form of work and heat, for example) which has to be supplied to the solid to bring it to the given configuration from the standard configuration.

We now fix our attention on a small element (or particle) of our solid body surrounding the point P, and of mass dm.† We denote the energy of the particle of mass dm at P

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* I originally worked out the theory for the case of body forces (*e.g.*, gravitational field) and body moments (*e.g.*, magnetised body in a magnetic field), in which it is also necessary to consider an initial stress. The consequent complications are of no more particular interest in a theory of visco-elasticity than in the theory of perfect elasticity ; and the equations of motion and expressions for the stress at the free surface of a solid, obtained in the case of *no* field of force, still hold—as approximations—in the case of a field of force and initial stress.

† In LOVE's " Elasticity " an element of certain volume in the standard configuration is considered for purposes of energy definition. Though physically the same, slight mathematical differences result. The fact that the surface integral on p. 93 is taken over the surface of the solid in the standard configuration implies a definition of stress with reference to the standard configuration, which prevents the stress-components from being components of a tensor. (All references to LOVE's " Elasticity " refer to 4th ed. (Cambridge, 1927).)

by $(E + E_0)\,dm$, where $E_0 dm$ is its energy in the standard configuration. Provided that the dimensions of the element are large compared with molecular dimensions, we can split the energy up into kinetic energy (of molar motion) and the remainder, which we call the non-kinetic energy. Hence

$$(E + E_0)\,dm = T dm + (U + U_0)\,dm,$$

where T denotes the kinetic energy per unit mass and $U + U_0$ the non-kinetic energy per unit mass in the configuration at time t. There is no kinetic energy in the standard configuration, by definition; and U_0 is the non-kinetic energy per unit mass in the standard configuration. The non-kinetic energy $(U + U_0)\,dm$ is independent of the motion of the solid.

The kinetic energy per unit mass, T, is given by the formula

$$T = \tfrac{1}{2} \sum_{\alpha=1,2,3} \left(\frac{\partial u_\alpha}{\partial t}\right)^2, \qquad (1\cdot31)$$

where $\partial u_\alpha/\partial t$ is the α-component of velocity of the element P.

Now, if the solid always returns to the standard configuration when released from a state of strain (it is supposed that a flow of heat is arranged to bring the solid to a uniform temperature θ_0), the physical properties of the element at P appear to be completely determined by the shape and temperature of the element. For there is no method of distinguishing between two configurations in which the shape and temperature of the typical element at P are identical. The shape of the element is completely determined by the strain tensor $\mathbf{e}$, which is, in turn, determined by its nine components $e_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$, only six of which are distinct on account of the relations $e_{\alpha\beta} = e_{\beta\alpha}$.

We suppose that no permanent set is possible, so that

$$U \equiv U(\theta, \mathbf{e}). \qquad (1\cdot32)$$

Unless the solid is uniform the expression for the non-kinetic energy will vary from one element to another of the body. Any element is determined by its position in the standard configuration, *i.e.*, by the vector $\mathbf{x}$. Thus for a non-uniform solid, we have*

$$U \equiv U(\theta, \mathbf{e}, \mathbf{x}). \qquad (1\cdot33)$$

In adopting this expression for the non-kinetic energy, we are neglecting the effects of plasticity; and strictly, the subsequent theory is limited to solids strained within the true limit of perfect (statical) elasticity. It is still possible, however, that the theory may represent approximately the behaviour of solids when the plastic effects are not entirely absent, and particularly solids strained only within the *practically defined* limit of perfect (statical) elasticity. For even in a plastic solid, besides the imperfectness

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* In the case of body forces and body moments due to a field of force, the functional form of U is given by $U \equiv U(\theta, \mathbf{e}, \mathbf{u}, \varpi, \mathbf{x})$.

of elasticity due to plasticity there may also be an independent dynamical imperfectness of elasticity, which is the dominant factor in certain cases, *e.g.*, rapid vibrations.*

1·4. *Change of configuration.*

We now consider a virtual change of configuration of a portion of the solid contained in an *arbitrary* closed surface, lying entirely within the solid body. This arbitrary closed surface is taken to move with the solid, *i.e.*, it always contains the same set of elements, being fixed in the solid but not fixed in space.

In the initial configuration, the temperature of the element at P is θ, and the displacement vector at P is $\mathbf{u}$. In the final configuration the temperature at P is $\theta + \delta\theta$, and the displacement vector $\mathbf{u} + \delta\mathbf{u}$. The change takes place in a time δt; and the solid is supposed to be both strained and straining in both the initial and final configurations. The increase in energy of the element, of mass dm at P, is

$$\delta\,[\mathrm{T}\,dm + (\mathrm{U} + \mathrm{U}_0)\,dm]$$
$$= (\delta\mathrm{T} + \delta\mathrm{U})\,dm.$$

Neither the mass nor the number of the elements changes during this change of configuration, so that the increase of energy of the portion of the solid under consideration is

$$\Sigma\,(\delta\mathrm{T} + \delta\mathrm{U})\,dm,$$

the summation extending over all the elements of which the portion of the solid is composed.

Now

$$dm = \rho\,d\tau,$$

where ρ is the density and $d\tau$ the volume of the element of mass dm.

Hence the increase in energy is

$$\iiint (\delta\mathrm{T} + \delta\mathrm{U})\,\rho\,d\tau, \quad \ldots\ldots\ldots\ldots \quad (1\cdot41)$$

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* *First Note on Plasticity.*—If the solid is *plastic*, a permanent set will remain after release from a state of strain; and the amount of the permanent set will depend on the previous strain history. Thus, the physical properties of the typical element at P are distinguished, not only by the strain and temperature of the element, but also by the permanent set it would acquire when released from the state of strain. (The effect of the strain history on the physical state is possibly due to irreversible molecular slipping or slipping at the interfaces of very small crystals contained within the element.)

When a solid receives a permanent set, the permanent set configuration has many of the properties of the old standard configuration. It seems that a plausible theory of plasticity could be worked out by putting $\mathbf{e} - \mathbf{e}'$ for $\mathbf{e}$ in the expression for the non-kinetic energy, where $\mathbf{e}'$ is the permanent set that the element would acquire if released from the instantaneous strained configuration. For in the plastic solid the "*physical strain*" is not measured by the *geometrical strain* $\mathbf{e}$, but (I would suggest) by the strain $\mathbf{e} - \mathbf{e}'$. Within the practically defined limit of perfect (statical) elasticity, $\mathbf{e}'$ must be less than $0\cdot0003\,\mathbf{e}$. I hope to deal with the effects of plasticity on these lines in a separate paper.

the volume integral being taken over the volume of the arbitrary portion of the solid at time t.

From (1·31) we obtain

$$\delta \mathrm{T} = \frac{\partial u_\alpha}{\partial t}\frac{\partial^2 u_\alpha}{\partial t^2}\,\delta t = f_\alpha\,\delta u_\alpha, \quad \ldots\ldots\ldots\ldots \quad (1{\cdot}42)$$

where $f_\alpha = \partial^2 u_\alpha/\partial t^2$ is the acceleration of the element at time t.

Now

$$\delta \mathrm{U} = \frac{\partial \mathrm{U}}{\partial \theta}\,\delta\theta + \frac{\partial \mathrm{U}}{\partial e_{\alpha\beta}}\,\delta e_{\alpha\beta}. \quad \ldots\ldots\ldots\ldots \quad (1{\cdot}43)$$

A convention is necessary regarding the differential coefficient $\partial\mathrm{U}/\partial e_{\alpha\beta}$. This is calculated with the conventions that $e_{\alpha\beta}$ and $e_{\beta\alpha}$ are to be treated as independent variables for the purposes of this differentiation, and that U is expressed symmetrically in $e_{\alpha\beta}$ and $e_{\beta\alpha}$. Thus $\partial\mathrm{U}/\partial e_{\alpha\beta} = \partial\mathrm{U}/\partial e_{\beta\alpha}$.

The first law of thermodynamics tells us that the increase in energy of the portion of the solid under consideration is equal to the total energy supplied, *i.e.*, the sum of the mechanical work done on the portion of the solid, and the heat energy supplied. Denoting by $\delta\mathrm{W}'$ the mechanical work done, and by $\delta\mathrm{Q}'d\tau$ the heat energy absorbed by the element of mass dm and volume $d\tau$ (the primes are used, in these cases, to indicate that $\delta\mathrm{W}'$, $\delta\mathrm{Q}'$ are not necessarily perfect differentials), the mathematical expression of the application of the first law becomes

$$\iiint (\delta\mathrm{T} + \delta\mathrm{U})\,\rho\,d\tau = \delta\mathrm{W}' + \iiint \delta\mathrm{Q}'\,d\tau. \quad \ldots\ldots\ldots \quad (1{\cdot}44)$$

The mechanical work done is simply the work done by the stresses across the surface of the portion of the body under consideration. Thus

$$\delta\mathrm{W}' = \iint p_{\nu\alpha}\,\delta u_\alpha\,d\mathrm{S}$$

where ν indicates the direction of the normal to the element of surface area $d\mathrm{S}$, and the integral is taken over the surface at time t.

Hence, using GREEN'S transformation,

$$\delta\mathrm{W}' = \iiint \frac{\partial}{\partial x_\beta}(p_{\beta\alpha}\,\delta u_\alpha)\,d\tau$$

$$= \iiint \left(\delta u_\alpha \frac{\partial}{\partial x_\beta}p_{\beta\alpha} + p_{\beta\alpha}\frac{\partial}{\partial x_\beta}\delta u_\alpha\right) d\tau.$$

But, from (1·23) and (1·22),

$$\frac{\partial}{\partial x_\beta}\,\delta u_\alpha = \delta e_{\alpha\beta} - \mathrm{E}_{\alpha\beta\gamma}\,\delta\varpi_\gamma,$$

so that

$$\delta W' = \iiint \left[\delta u_\alpha \frac{\partial}{\partial x_\beta} p_{\beta\alpha} + p_{\beta\alpha} \left(\delta e_{\alpha\beta} - E_{\alpha\beta\gamma} \, \delta \varpi_\gamma \right) \right] d\tau. \quad . \quad . \quad . \quad (1\cdot45)$$

By virtue of this expression for $\delta W'$, the relation $(1\cdot44)$ becomes

$$\iiint (\delta T + \delta U) \, \rho \, d\tau$$

$$= \iiint \left[\delta u_\alpha \frac{\partial}{\partial x_\beta} p_{\beta\alpha} + p_{\beta\alpha} \left(\delta e_{\alpha\beta} - E_{\alpha\beta\gamma} \, \delta \varpi_\gamma \right) \right] d\tau + \iiint \delta Q' \, d\tau. \quad . \quad (1\cdot46)$$

Now the volume integral in each term of $(1\cdot46)$ is taken over the volume occupied by the arbitrary portion of the solid at time t. Since the portion of the solid is arbitrary, the volume of integration is arbitrary, and the equation $(1\cdot46)$ must hold for the integrands. Hence

$$(\delta T + \delta U) \, \rho = \frac{\partial p_{\beta\alpha}}{\partial x_\beta} \delta u_\alpha + p_{\beta\alpha} \, \delta e_{\alpha\beta} - p_{\beta\alpha} \, E_{\alpha\beta\gamma} \, \delta \varpi_\gamma + \delta Q'. \quad . \quad . \quad . \quad (1\cdot47)$$

Using the expressions $(1\cdot42)$ and $(1\cdot43)$ for δT and δU, we obtain

$$\delta u_\alpha \left(\rho f_\alpha - \frac{\partial}{\partial x_\beta} p_{\beta\alpha} \right) + E_{\alpha\beta\gamma} p_{\beta\alpha} \, \delta \varpi_\gamma$$

$$+ \left(\rho \frac{\partial U}{\partial e_{\alpha\beta}} - p_{\beta\alpha} \right) \delta e_{\alpha\beta} + \rho \frac{\partial U}{\partial \theta} \, \delta \theta - \delta Q' = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1\cdot48)$$

Any results that may be obtained from this equation must depend on the expression for the quantity of heat $\delta Q'$, and this depends on the physical character of the change of configuration.

1·5. *Reversible and irreversible changes of configuration.*

The amount of heat energy required to raise the temperature of the element P, of mass dm (and volume $d\tau$), by an amount $\delta\theta$ without altering its shape is $c_v \, \delta\theta \, dm$, where c_v is the specific heat at constant strain—analogous to the specific heat at constant volume for a gas. Now the physics of a reversible change is completely determined by the initial and final configurations. Thus, if the change of configuration is a reversible change, the amount of heat which must be supplied to the element P to increase the temperature by an amount $\delta\theta$ and at the same time to alter the shape according to a change of strain $\delta \mathbf{e}$ is given by a formula of the type

$$\delta Q' \, d\tau = (q_{\alpha\beta} \, \delta e_{\alpha\beta} + c_v \, \delta\theta) \, dm, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1\cdot51)$$

where $q_{\alpha\beta}$ is a component of a second order tensor. We still regard $e_{\alpha\beta}$ and $e_{\beta\alpha}$ as distinct, and take $q_{\alpha\beta} = q_{\beta\alpha}$ without loss of generality.

In an *irreversible* change the physics of the change depends not only on the initial and final configurations, but also on the manner in which the change is made. It is convenient to take the second law of thermodynamics in the form enunicated by KELVIN, viz., "There is in the material world a tendency to the dissipation of mechanical energy."* We suppose that a quantity of energy, $2F\,\delta t$ per unit volume, is dissipated during the change of configuration of the solid. It immediately follows from KELVIN's statement of the second law that the "dissipation function," F, is essentially positive.

Since a quantity of energy, $2F\,\delta t$ per unit volume, is dissipated, the amount of mechanical work necessary to effect the change of configuration is greater by an amount $2F\,\delta t$ per unit volume than the quantity of work which would be necessary if the change of configuration were a reversible change. But from the first law, the sum of mechanical work and heat energy which has to be supplied to effect the change is equal to the increase of energy during the change. Thus in an irreversible change the quantity of heat energy which must be supplied per unit volume is less by an amount $2F\,\delta t$ than if the change were reversible. Thus, using (1·51), we have

$$\delta Q'\,d\tau = (q_{\alpha\beta}\,\delta e_{\alpha\beta} + c_v\,\delta\theta)\,dm - 2F\,\delta t\,d\tau$$

in an irreversible change.

Since $dm = \rho d\tau$, this reduces to

$$\delta Q' = \rho\,(q_{\alpha\beta}\,\delta e_{\alpha\beta} + c_v\,\delta\theta) - 2F\,\delta t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1\cdot52)$$

Substituting this value of $\delta Q'$ in the equation (1·48), we obtain

$$\delta u_\alpha\left(\rho f_\alpha - \frac{\partial}{\partial x_\beta}\,p_{\beta\alpha}\right) + E_{\alpha\beta\gamma}p_{\beta\alpha}\,\delta\varpi_\gamma$$
$$+\left[\rho\left(\frac{\partial U}{\partial e_{\alpha\beta}} - q_{\alpha\beta}\right) - p_{\beta\alpha}\right]\delta e_{\alpha\beta} + \rho\left(\frac{\partial U}{\partial\theta} - c_v\right)\delta\theta + 2F\,\delta t = 0. \quad . \quad . \quad . \qquad (1\cdot53)$$

1·6. *The dissipation function: definition of a "visco-elastic solid."*

So far, apart from the expression assumed for the non-kinetic energy which neglects effects due to plasticity, we have merely developed the thermodynamics of a continuous medium in accordance with the first and second laws of thermodynamics. To make any further progress it is necessary to choose some expression for the dissipation function. If the solid is perfectly elastic, changes of configuration are reversible and the dissipation function is identically equal to zero. We suppose that, for the solid under consideration, changes of configuration are characterized by the following properties :—

(1) In the limiting case when a change of geometrical configuration is made indefinitely slowly, the change is reversible and there is no dissipation of energy. Any work done in causing such a change is wholly recoverable as mechanical work.

(2) When a change of geometrical configuration is made at finite speed, the change is irreversible, and dissipation of energy occurs. Work done in causing such a change is not wholly recoverable as mechanical work.

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* Sir W. THOMSON. 'Collected Papers,' vol. 5, p. 11.

A solid, the change of whose configuration is characterised by the above properties,* is defined as a *visco-elastic solid*. It is clear from assumption (1) that a visco-elastic solid behaves statically as a perfectly elastic solid.

According to assumption (2) above the dissipation function F depends on the value of the *rate of strain tensor* $\partial \mathbf{e}/\partial t = \dot{\mathbf{e}}$; though it may depend on other factors as well *e.g.*, the strain tensor $\mathbf{e}$, and the temperature θ.

Expanding F in powers of the components of the rate of strain tensor by the generalised TAYLOR's theorem, we obtain

$$\mathrm{F} = (\mathrm{F})_{\dot{\mathbf{e}}=0} + \left(\frac{\partial \mathrm{F}}{\partial \dot{e}_{\alpha\beta}}\right)_{\dot{\mathbf{e}}=0} \dot{e}_{\alpha\beta} + \tfrac{1}{2}\left(\frac{\partial^2 \mathrm{F}}{\partial \dot{e}_{\alpha\beta}\,\partial \dot{e}_{mn}}\right)_{\dot{\mathbf{e}}=0} \dot{e}_{\alpha\beta}\,\dot{e}_{mn} + \text{higher powers.} \qquad (1\cdot 61)$$

We regard $\dot{e}_{\alpha\beta}$ and $\dot{e}_{\beta\alpha}$ as distinct in the differentiation of the function F; and assume, without loss of generality, that F is expressed symmetrically in terms of $\dot{e}_{\alpha\beta}$ and $\dot{e}_{\beta\alpha}$.

But, by assumption (1), $2\mathrm{F}\,\delta t \to 0$ as $\dot{\mathbf{e}} \to 0$.

Now

$$\dot{e}_{\alpha\beta}\,\delta t = \frac{\partial e_{\alpha\beta}}{\partial t}\,\delta t = \delta e_{\alpha\beta} \neq 0.$$

Hence

and

$$\left.\begin{aligned}(\mathrm{F})_{\dot{\mathbf{e}}=0} &= 0\\ \left(\frac{\partial \mathrm{F}}{\partial \dot{e}_{\alpha\beta}}\right)_{\dot{\mathbf{e}}=0} &= 0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1\cdot 62)$$

We further suppose that the speed of deformation is so small that we can neglect powers of the rate of strain components of order higher than the second. The expression (1·61) for F then becomes

$$\begin{aligned}2\mathrm{F} &= \left(\frac{\partial^2 \mathrm{F}}{\partial \dot{e}_{\alpha\beta}\,\partial \dot{e}_{mn}}\right)_{\dot{\mathbf{e}}=0} \dot{e}_{\alpha\beta}\,\dot{e}_{mn}\\ &= a_{\alpha\beta mn}\,\dot{e}_{\alpha\beta}\,\dot{e}_{mn}, \quad \ldots\ldots\ldots\ldots \quad (1\cdot 63)\end{aligned}$$

where

$$a_{\alpha\beta mn} = \left(\frac{\partial^2 \mathrm{F}}{\partial \dot{e}_{\alpha\beta}\,\partial \dot{e}_{mn}}\right)_{\dot{\mathbf{e}}=0}.$$

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* *Second Note on Plasticity.*—If the solid is plastic assumption (1) has to be modified to agree with the modified form of the expression for the non-kinetic energy. For in a plastic solid, even in a fixed geometrical configuration, an irreversible process is going on—possibly, as suggested in the first note, some sort of slipping within the element.

A new assumption has been made in (1), over and above the one that the solid is not plastic. For though the solid would be plastic if an irreversible process goes on when there is no change of geometrical configuration, it would not necessarily be plastic if an irreversible process occurs in an infinitely slow change of geometrical configuration.

The coefficients $a_{\alpha\beta mn}$, which we call the "*viscous coefficients*," are components of a tensor of the fourth order in three dimensions. There are, in general, 81 components of such a tensor, but in this case the number is reduced by symmetry relations. For $a_{\alpha\beta mn}$ is unaltered if α and β or m and n are interchanged, on account of the convention that F is expressed symmetrically in terms of $e_{\alpha\beta}$ and $e_{\beta\alpha}$. It is also unaltered if $\alpha\beta$ are interchanged with mn, on account of the assumed commutative property of the operators $\partial/\partial\dot{e}_{\alpha\beta}$ and $\partial/\partial\dot{e}_{mn}$ acting on F. (This assumption implies only the continuity of the physical properties of the solid through varied rates of straining.) The number of distinct viscous coefficients—the components $a_{\alpha\beta mn}$—is thus 21 in the general case of an æolotropic solid.

Since F is now expressed (approximately) as a polynomial of the second order in the rate of strain components, we have, using Euler's theorem,

$$2\mathrm{F} = \frac{\partial \mathrm{F}}{\delta \dot{e}_{\alpha\beta}}\,\dot{e}_{\alpha\beta}.$$

Hence

$$2\mathrm{F}\,\delta t = \frac{\partial \mathrm{F}}{\partial \dot{e}_{\alpha\beta}}\,\delta e_{\alpha\beta}, \quad \ldots\ldots\ldots\ldots \quad (1\cdot64)$$

where

$$\frac{\partial \mathrm{F}}{\partial \dot{e}_{\alpha\beta}} = a_{\alpha\beta mn}\,\dot{e}_{mn}. \quad \ldots\ldots\ldots\ldots \quad (1\cdot65)$$

We remark that the dissipation function is of the same form as for a viscous fluid, and note also its similarity to the Rayleigh dissipation function which occurs in the theory of a dissipative mechanical system.

"*Visco-elastic limits.*"—We are now in a position to see the sort of limits within which the resulting theory may give an approximately accurate representation of the dynamical behaviour of solids. If plastic effects are absent and assumption (1) above concerning the physical character of changes of configuration is correct, then the theory is limited only by the limit of the validity of the expression of the dissipation function F in its approximate form (1·63). An "*upper visco-elastic limit*" could therefore be *practically* defined such that the theory of visco-elasticity is only applicable to solids when the rate of strain does not exceed this limiting value.

If plastic effects are not absent and (or) the assumption (1) is incorrect, then the dissipation function is not expressed correctly. The inaccuracy due to this faulty expression is clearly most important for slow rates of strain, but may be unimportant for higher rates of strain when the dissipation function defined above becomes greater. Thus, even if the solid is plastic, it is possible that the theory of visco-elasticity may represent approximately its dynamical behaviour, provided that the rate of strain *exceeds* a *practically* defined limiting value, which we call the "*lower visco-elastic limit.*" The theory is then applicable for rates of strain in between the upper and lower visco-elastic limits. The determination of the limits is a matter for experiment, but it seems probable that for the more plastic solids the lower visco-elastic limit may be greater than the upper visco-elastic limit, so that the theory of visco-elasticity would have no range of applicability to such solids.

1·7. *The equations of motion and the stress-strain relations in a visco-elastic solid.*

We insert the value of $2\mathrm{F}\delta t$ obtained in (1·64) into the equation (1·53), to obtain

$$\delta u_{\alpha}\left(\rho f_{\alpha}-\frac{\partial}{\partial x_{\beta}}\,p_{\beta\alpha}\right)+\mathrm{E}_{\alpha\beta\gamma}p_{\beta\alpha}\,\delta\varpi_{\gamma}$$
$$+\left[\rho\left(\frac{\partial \mathrm{U}}{\partial e_{\alpha\beta}}-q_{\alpha\beta}\right)+\frac{\partial \mathrm{F}}{\partial \dot{e}_{\alpha\beta}}-p_{\beta\alpha}\right]\delta e_{\alpha\beta}+\rho\left(\frac{\partial \mathrm{U}}{\partial \theta}-c_{v}\right)\delta\theta=0. \qquad (1\cdot71)$$

The variations δu_{α}, $\delta\varpi_{\gamma}$, $\delta e_{\alpha\beta}$, $\delta\theta$ in this relation are the variations of the values of u_{α}, ϖ_{γ}, $e_{\alpha\beta}$, θ at the point P in the virtual change of configuration which has been considered. Though $e_{\alpha\beta}$ and ϖ_{γ} are dependent on the *gradient* of u_{α}, it is readily seen that (apart from the fact that $e_{\alpha\beta}=e_{\beta\alpha}$) there exist a set of virtual changes of configuration in which any one *alone* of all the variations involved in (1·71) is different from its value in the change of configuration already considered.

Suppose, for instance, that the variations δu_{α} in the virtual change already considered have values $\delta u_{1}\,(x_{1}\,x_{2}\,x_{3})$, $\delta u_{2}\,(x_{1}\,x_{2}\,x_{3})$, $\delta u_{3}\,(x_{1}\,x_{2}\,x_{3})$, throughout the solid, $\delta e_{\alpha\beta}$ and $\delta\varpi_{\gamma}$ being determined from the gradient of δu_{1}, δu_{2}, δu_{3}, according to the formulæ (1·22) and (1·23). Let us denote the values of the variations in the above virtual change by the superscript 0, and the current co-ordinates by X_{1}, X_{2}, X_{3} while the co-ordinates of P are x_{1}, x_{2}, x_{3}.

In a virtual change in which the displacement variations are

$$(\delta u_{1})^{0}+k, \quad (\delta u_{2})^{0}, \quad (\delta u_{3})^{0},$$

of all the variations δu_{1} alone is different at P.

In a virtual change in which the displacement variations are

$$(\delta u_{1})^{0}+l\,(\mathrm{X}_{1}-x_{1}), \quad (\delta u_{2})^{0}, \quad (\delta u_{3})^{0},$$

of all the variations e_{11} alone is different at P.

In a virtual change in which the displacement variations are

$$(\delta u_{1})^{0}, \quad (\delta u_{2})^{0}+m\,(\mathrm{X}_{3}-x_{3}), \quad (\delta u_{3})^{0}+m\,(\mathrm{X}_{2}-x_{2}),$$

of all the variations e_{23} and e_{32} alone are different at P.

The enumeration of the particulars virtual changes in which all except one of the individual variations remain the same (with the exception that $e_{\alpha\beta}=e_{\beta\alpha}$) is obvious.

Hence we have a set of equations of the type (1·71), from which it is found that the coefficients of u_α, ϖ_γ, θ, $(e_{\alpha\beta}+e_{\beta\alpha})$ all vanish. Thus

$$\rho f_\alpha = \frac{\partial}{\partial x_\beta} p_{\beta\alpha} \quad \ldots \ldots \ldots \quad (1\cdot72)$$

$$\mathrm{E}_{\alpha\beta\gamma}\, p_{\beta\alpha} = 0 \quad \ldots \ldots \ldots \quad (1\cdot73)$$

$$\begin{aligned} p_{\alpha\beta} + p_{\beta\alpha} &= \left[\rho\left(\frac{\partial \mathrm{U}}{\partial e_{\alpha\beta}} - q_{\alpha\beta}\right) + \frac{\partial \mathrm{F}}{\partial \dot{e}_{\alpha\beta}}\right] + \left[\rho\left(\frac{\partial \mathrm{U}}{\partial e_{\beta\alpha}} - q_{\beta\alpha}\right) + \frac{\partial \mathrm{F}}{\partial \dot{e}_{\beta\alpha}}\right] \\ &= 2\left[\rho\left(\frac{\partial \mathrm{U}}{\partial e_{\alpha\beta}} - q_{\alpha\beta}\right) + \frac{\partial \mathrm{F}}{\partial \dot{e}_{\alpha\beta}}\right] \quad \ldots \ldots \ldots \quad (1\cdot74) \end{aligned}$$

$$\frac{\partial \mathrm{U}}{\partial \theta} = c_v. \quad \ldots \ldots \ldots \quad (1\cdot75)$$

From (1·73) we have $p_{\alpha\beta} = p_{\beta\alpha}$ when α is not equal to β, so that the stress tensor is a symmetrical tensor.

Thus (1·74) reduces to

$$p_{\alpha\beta} = \rho\left(\frac{\partial \mathrm{U}}{\partial e_{\alpha\beta}} - q_{\alpha\beta}\right) + \frac{\partial \mathrm{F}}{\partial \dot{e}_{\alpha\beta}}, \quad \ldots \ldots \ldots \quad (1\cdot76)$$

which are the stress-strain relations in a visco-elastic solid.

We note that the complete application of the principle of virtual work has given the equations of motion (1·72), the symmetry property of the stress tensor (1·73), and the stress-strain relations (1·76) as the result of one calculation.

In the case of a perfectly elastic solid the dissipation function vanishes identically so that the stress-strain relations (1·76) become*

$$p_{\alpha\beta} = \rho\left(\frac{\partial \mathrm{U}}{\partial e_{\alpha\beta}} - q_{\alpha\beta}\right).$$

These stress-strain relations for a perfectly elastic solid can, of course, be reduced to HOOKE's law as an approximation, though by keeping a more accurate form it is possible to discuss thermo-elastic relations. We now reduce the "elastic terms" in the stress-strain relations (1·76) to a form which will enable us to investigate the thermo-elastic effects in a visco-elastic solid.

It is first necessary to obtain formulæ for the thermal coefficients $q_{\alpha\beta}$. We consider a change of configuration of an element of the solid of mass *dm* and volume $d\tau$ such

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* An expression for the stress in a perfectly elastic solid similar to this is obtained by JEFFREYS in the chapter on Elasticity in his "Cartesian Tensors" (Camb., 1931). The density ρ does not occur in his expression; its presence here is due to the unusual definition of the energy per unit mass—see footnote † p. 353. Apart from this difference, the remaining results in this section (1·7) are similar to those obtained by JEFFREYS.

that there is no displacement or rotation of the element. The change is also effected infinitely slowly, so that the change is reversible, and the kinetic energy is always zero. The expression (1·76) for the stress during this change becomes simply

$$p_{\alpha\beta} = \rho \left(\frac{\partial \mathrm{U}}{\partial e_{\alpha\beta}} - q_{\alpha\beta}\right). \quad \ldots \ldots \ldots \ldots \quad (1{\cdot}770)$$

Inserting the value for $\delta \mathrm{Q}'$ in a reversible change, as given by (1·51), into the equation (1·47) for $\delta \mathrm{U}$, we have

$$\delta \mathrm{U} = \left(\frac{p_{\beta\alpha}}{\rho} + q_{\alpha\beta}\right) \delta e_{\alpha\beta} + c_v \delta \theta \; ; \quad \ldots \ldots \ldots \quad (1{\cdot}771)$$

and this is a perfect differential.

Since the change is reversible, the increase of entropy of the element is a perfect differential. But

$$\delta \mathrm{S} = \frac{\delta \mathrm{Q}' \, d\tau}{\theta},$$

so that

$$\frac{q_{\alpha\beta}}{\theta} \delta e_{\alpha\beta} + \frac{c_v}{\theta} \delta \theta \quad \ldots \ldots \ldots \ldots \quad (1{\cdot}772)$$

is a perfect differential.

If further the change is isothermal, we see from the fact that (1·771) and (1·772) are perfect differentials that $\frac{p_{\beta\alpha}}{\rho} \delta e_{\alpha\beta}$ is a perfect differential. Hence, there is a function W such that

$$\left(\frac{\partial \mathrm{U}}{\partial e_{\alpha\beta}} - q_{\alpha\beta}\right) = \frac{p_{\alpha\beta}}{\rho} = \frac{p_{\beta\alpha}}{\rho} = \frac{\partial \mathrm{W}}{\partial e_{\alpha\beta}}. \quad \ldots \ldots \ldots \quad (1{\cdot}773)$$

Again, $e_{\alpha\beta}$ and $e_{\beta\alpha}$ are regarded as formally distinct in the differentiation of W, and we assume that W is expressed symmetrically in terms of $e_{\alpha\beta}$ and $e_{\beta\alpha}$.

Writing down the conditions that (1·771) and (1·772) are perfect differentials and eliminating c_v, we find that

$$q_{\alpha\beta} = -\, \theta \frac{\partial}{\partial \theta} \left(\frac{\partial \mathrm{W}}{\partial e_{\alpha\beta}}\right). \quad \ldots \ldots \ldots \ldots \quad (1{\cdot}774)$$

By virtue of the relations (1·773), the stress-strain relations (1·76) for a visco-elastic solid, reduce to the form

$$p_{\alpha\beta} = \rho \frac{\partial \mathrm{W}}{\partial e_{\alpha\beta}} + \frac{\partial \mathrm{F}}{\partial \dot{e}_{\alpha\beta}}. \quad \ldots \ldots \ldots \ldots \quad (1{\cdot}775)$$

The function W is a function of the temperature θ, and the strain e. Expanding it in powers of the strain components by Taylor's Theorem, we have

$$\mathrm{W} = c' + c'_{\alpha\beta} e_{\alpha\beta} + \tfrac{1}{2} c'_{\alpha\beta mn} e_{\alpha\beta} e_{mn} + \text{higher powers}, \quad \ldots \ldots \quad (1{\cdot}780)$$

where

$$c' = (\mathrm{W})_{\mathrm{e}=0} \, ; \; c'_{\alpha\beta} = \left(\frac{\partial \mathrm{W}}{\partial e_{\alpha\beta}}\right)_{\mathrm{e}=0} ; \; c'_{\alpha\beta mn} = \left(\frac{\partial^2 \mathrm{W}}{\partial e_{\alpha\beta} \theta e_{mn}}\right)_{\mathrm{e}=0}. \quad \ldots \ldots \quad (1{\cdot}781)$$

Thus

$$\frac{\partial W}{\partial e_{\alpha\beta}} = c'_{\alpha\beta} + c'_{\alpha\beta mn} e_{mn} + \text{higher powers}\,; \quad \ldots\ldots \quad (1\cdot782)$$

and, supposing that the strain is so small that powers of the strain components of order higher than the first can be neglected in the expression for the stress, and using the formula (1·65) for $\dfrac{\partial F}{\partial \dot{e}_{\alpha\beta}}$, the stress-strain relations (1·775) for a visco-elastic solid further reduce to

$$p_{\alpha\beta} = \rho c'_{\alpha\beta} + \rho c'_{\alpha\beta mn} e_{mn} + a_{\alpha\beta mn} \dot{e}_{mn}. \quad \ldots\ldots\ldots \quad (1\cdot790)$$

The density ρ (at time t) depends on the strain. Its value is given by

$$\rho\,(1 + \Delta) = \rho_0, \quad \ldots\ldots\ldots\ldots \quad (1\cdot791)$$

where ρ_0 is the density in the standard configuration, and Δ is the dilatation. The dilatation is expressed in terms of the strain components by the well-known formula

$$\Delta = 1 + U_{mn} e_{mn}, \quad \ldots\ldots\ldots\ldots \quad (1\cdot792)$$

U_{mn} being a component of the second order cartesian substitution tensor. (Care should be taken to avoid confusion with the non-kinetic energy function U.)

1·8. *The thermo-elastic relations in a visco-elastic solid.*

The coefficients $c'_{\alpha\beta}$, $c'_{\alpha\beta mn}$ depend on the temperature, and the viscous coefficient $a_{\alpha\beta mn}$ may depend on both the temperature and the strain. Thus the value of the stress in a visco-elastic solid, given by the stress-strain relations (1·790), depends on the temperature θ, which is determined by the initial temperature and the thermal character of the subsequent changes of configuration. We first express $c'_{\alpha\beta}$ in terms of $c'_{\alpha\beta mn}$ and the coefficients of thermal expansion.

Suppose the solid is in its unstrained standard configuration at temperature θ_0, and that its temperature is raised to θ without stress. The resulting thermal expansion causes a displacement represented by the equation

$$u_\alpha = (\theta - \theta_0)\, A_{\alpha\beta} x_\beta, \ldots\ldots\ldots\ldots \quad (1\cdot810)$$

where $A_{\alpha\beta}$ is a component of a second-order tensor.

The resulting strain is given by

$$e_{\alpha\beta} = \tfrac{1}{2}\,(A_{\alpha\beta} + A_{\beta\alpha})\,(\theta - \theta_0) = B_{\alpha\beta},\,(\theta - \theta_0), \quad \ldots\ldots \quad (1\cdot811)$$

where $B_{\alpha\beta}$ is a component of a symmetrical tensor.

Since, by hypothesis, the expansion took place under no stress the stress-strain relations (1·790) give

$$0 = \rho\,\{c'_{\alpha\beta} + c'_{\alpha\beta mn} B_{mn}\ (\theta - \theta_0)\}.$$

Thus*

$$c'_{\alpha\beta} = -c'_{\alpha\beta mn}B_{mn}(\theta - \theta_0), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1 \cdot 812)$$

and the stress-strain relations (1·790) reduce to

$$p_{\alpha\beta} = \rho c'_{\alpha\beta mn}[e_{mn} - B_{mn}(\theta - \theta_0)] + a_{\alpha\beta mn}\dot{e}_{mn}. \quad . \; . \; . \; . \; . \quad (1 \cdot 813)$$

Now powers of the strain components of order higher than the second have already been neglected in obtaining the expression (1·790) for the stress components. Keeping to the same order of approximation, we replace ρ by its constant value in the standard configuration, *i.e.*, ρ_0. Putting

$$\rho_0 c'_{\alpha\beta mn} = c_{\alpha\beta mn}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1 \cdot 814)$$

the stress-strain relations for a visco-elastic solid become

$$p_{\alpha\beta} = c_{\alpha\beta mn}[e_{mn} - B_{mn}(\theta - \theta_0)] + a_{\alpha\beta mn}\dot{e}_{mn}. \quad . \; . \; . \; . \; . \quad (1 \cdot 820)$$

The coefficients $c_{\alpha\beta mn}$ are the well-known *elastic coefficients* (or *constants*). They are subject to the same symmetry relations as the *viscous coefficients* $a_{\alpha\beta mn}$ (see § 1·6), and only 21 are distinct in the general case of an æolotropic solid.

When changes of configuration take place isothermally, the temperature θ is constant and equal to θ_0; and the stress-strain relations assume the simple form

$$p_{\alpha\beta} = c_{\alpha\beta mn}\,e_{mn} + a_{\alpha\beta mn}\,\dot{e}_{mn}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1 \cdot 821)$$

This is the same as the stress-strain relation suggested by VOIGT (see p. 346), with the exception that here only 21 of the viscous coefficients $a_{\alpha\beta mn}$ are distinct.

In the case of a non-isothermal change of configuration the change of temperature is given by equation (1·52), namely,

$$c_v\,\delta\theta = \frac{1}{\rho}(\delta Q' + 2F\,\delta t) - q_{\alpha\beta}\,\delta e_{\alpha\beta}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1 \cdot 830)$$

But we have the result (1·774) that

$$\begin{aligned} q_{\alpha\beta} &= -\theta\frac{\partial}{\partial\theta}\left(\frac{\partial W}{\partial e_{\alpha\beta}}\right) \\ &= \theta\,c'_{\alpha\beta mn}B_{mn} \quad \text{approximately}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1 \cdot 831) \end{aligned}$$

using the expression (1·782) for $\dfrac{\partial W}{\partial e_{\alpha\beta}}$, and (1·812) for $c'_{\alpha\beta}$.

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* In the case of initial stress, $c'_{\alpha\beta} = p^0_{\alpha\beta}/\rho_0$ when $\theta = \theta_0$. The presence of ρ does in this case make a difference to the expression for the stress.

Integrating equation (1·830), we obtain

$$c_v(\theta - \theta_0) = \int_0^t \frac{1}{\rho}(\delta Q' + 2F\,\delta t) - q_{\alpha\beta}e_{\alpha\beta}$$

approximately, as $q_{\alpha\beta}$, given by formula (1·831), is independent of the strain.

Changing the dummy suffixes in this result from α, β to m, n, we have

$$c_v(\theta - \theta_0) = \int_0^t \frac{1}{\rho}(\delta Q' + 2F\,\delta t) - q_{mn}e_{mn}. \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}832)$$

Now

$$c_{\alpha\beta mn}B_{mn} = \rho c'_{\alpha\beta mn}B_{mn}$$

$$= \frac{\rho q_{\alpha\beta}}{\theta} \qquad \text{by } (1{\cdot}831). \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}833)$$

Thus the general stress-strain relations (1·820) can be written

$$p_{\alpha\beta} = c_{\alpha\beta mn}e_{mn} - \frac{\rho q_{\alpha\beta}}{\theta}(\theta - \theta_0) + a_{\alpha\beta mn}\dot{e}_{mn}$$

$$= \left(c_{\alpha\beta mn} + \frac{\rho q_{\alpha\beta}q_{mn}}{\theta c_v}\right)e_{mn} + a_{\alpha\beta mn}\dot{e}_{mn} - \frac{\rho q_{\alpha\beta}}{\theta c_v}\int_0^t \frac{1}{\rho}(\delta Q' + 2F\,\delta t), \quad . \quad . \quad (1{\cdot}834)$$

using the expression (1·832) for $(\theta - \theta_0)$.

Thus, taking temperature changes into account, the stress-strain relations of a visco-elastic solid depend on the previous strain history on account of the term $-\frac{\rho q_{\alpha\beta}}{\theta c_v}\int \frac{1}{\rho}2F\,\delta t$.

For adiabatic changes $\delta Q' = 0$, and the stress-strain relations (1·834) reduce to

$$p_{\alpha\beta} = c''_{\alpha\beta mn}e_{mn} + a_{\alpha\beta mn}\dot{e}_{mn} - \frac{\rho q_{\alpha\beta}}{\theta c_v}\int_0^t \frac{1}{\rho}2F\,\delta t. \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}835)$$

Comparing this with the stress-strain relations (1·821) for isothermal changes, we see that the "adiabatic elastic constants" are

$$c''_{\alpha\beta mn} = c_{\alpha\beta mn} + \frac{\rho q_{\alpha\beta}q_{mn}}{\theta c_v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}836)$$

where, by (1·831),

$$q_{\alpha\beta} = \frac{\theta}{\rho}c_{\alpha\beta rs}B_{rs}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}837)$$

and the stress component $p_{\alpha\beta}$ is also reduced by an amount $\frac{\rho q_{\alpha\beta}}{\theta c_v}\int_0^t \frac{1}{\rho}2F\,\delta t$.

Formula (1·836) is the well-known formula for the adiabatic elastic constants.* The term $-\frac{\rho q_{\alpha\beta}}{\theta c_v}\int_0^t \frac{1}{\rho}2F\,\delta t$, due to the viscosity of the solid, is negative or zero in the case of an isotropic visco-elastic solid.

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* See, for instance, VOIGT, 'Gött. Nach.,' vol. 2, p. 174 (1900).

Changes of configuration of a solid are never really adiabatic, but $\delta Q'$ is given by the formula for the conduction of heat,

$$\delta Q' = \frac{\partial}{\partial x_\alpha}\left(K \frac{\partial \theta}{\partial x_\alpha}\right)\delta t,$$

where K is the thermal conductivity.

The stress-strain relations in their accurate form (1·834) lead to integro-differential equations of motion. The modifications due to thermo-elastic effects are not of any importance in problems concerning the vibrations of solids, though the integral term may be important in the discussion of afterworking effects. As an approximation, the stress-strain relations are taken in the simple form for isothermal changes, namely (1·821). The equations of motion and the stress-strain relations for a visco-elastic solid in this approximate form, differ from the corresponding equations for a perfectly elastic solid only in the fact that the elastic constants $c_{\alpha\beta mn}$ are replaced by *linear operators* $c_{\alpha\beta mn} + a_{\alpha\beta mn}\dfrac{\partial}{\partial t}$.

Condition for the stability of equilibrium of the standard configuration.—It is a matter of experience that a solid is in *stable* equilibrium in its standard configuration, as here defined. As we are neglecting thermo-elastic effects, we put

$$c'_{\alpha\beta} = 0, \quad q_{\alpha\beta} = 0, \quad \theta = \theta_0.$$

Thus (1·773) reduces to $\dfrac{\partial U}{\partial e_{\alpha\beta}} = \dfrac{\partial W}{\partial e_{\alpha\beta}}$; and we have $U = \frac{1}{2}c'_{\alpha\beta mn}e_{\alpha\beta}e_{mn}$.

The condition for stability (*necessary* as well as *sufficient* since there are dissipative forces)* is that the non-kinetic energy should be a minimum in the standard configuration, that is

$$\iiint \rho U \, d\tau > 0,$$

But

$$\begin{aligned}\rho U &= \tfrac{1}{2}\rho c'_{\alpha\beta mn}e_{\alpha\beta}e_{mn}\\ &= \tfrac{1}{2}c_{\alpha\beta mn}e_{\alpha\beta}e_{mn},\end{aligned}$$

neglecting powers of the strain components of order higher than the second.

The condition for stability of equilibrium is thus

$$\iiint c_{\alpha\beta mn}e_{\alpha\beta}e_{mn}d\tau > 0. \quad \ldots\ldots\ldots\ldots \quad (1{\cdot}840)$$

If, as we have already supposed, there is no initial stress, then *each element* of the solid is in stable equilibrium in the standard configuration. The condition for stability then becomes the well-known

$$c_{\alpha\beta mn}e_{\alpha\beta}e_{mn} > 0. \quad \ldots\ldots\ldots\ldots \quad (1{\cdot}841)$$

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* See LAMB'S "Higher Mechanics," p. 208.

1·9. *The isotropic visco-elastic solid.*

In an isotropic solid, the viscous and elastic coefficients, $a_{\alpha\beta mn}$ and $c_{\alpha\beta mn}$, must be *isotropic tensors*. We accordingly express $a_{\alpha\beta mn}$ and $c_{\alpha\beta mn}$ as the most general cartesian tensor of the fourth order in three dimensions.* The operator $c_{\alpha\beta mn} + a_{\alpha\beta mn}\dfrac{\partial}{\partial t}$ is thus given by a formula†

$$c_{\alpha\beta mn} + a_{\alpha\beta mn}\frac{\partial}{\partial t}$$

$$\equiv\left(\lambda + \lambda'\frac{\partial}{\partial t}\right)\mathrm{U}_{\alpha\beta}\mathrm{U}_{mn} + \left(\mu + \mu'\frac{\partial}{\partial t}\right)(\mathrm{U}_{\alpha m}\mathrm{U}_{\beta n} + \mathrm{U}_{\alpha n}\mathrm{U}_{\beta m}) + \left(v + v'\frac{\partial}{\partial t}\right) \times (\mathrm{U}_{\alpha m}\mathrm{U}_{\beta n} - \mathrm{U}_{\alpha n}\mathrm{U}_{\beta m}).$$

Hence, the stress-strain relations (1·821) become, for an isotropic solid,

$$p_{\alpha\beta} = \left(\lambda + \lambda'\frac{\partial}{\partial t}\right)\mathrm{U}_{\alpha\beta}\mathrm{U}_{mn}e_{mn} + \left(\mu + \mu'\frac{\partial}{\partial t}\right)(e_{\alpha\beta} + e_{\beta\alpha}) + \left(\nu + \nu'\frac{\partial}{\partial t}\right)(e_{\alpha\beta} - e_{\beta\alpha})$$

$$= \mathrm{U}_{\alpha\beta}\left(\lambda + \lambda'\frac{\partial}{\partial t}\right)\Delta + 2\left(\mu + \mu'\frac{\partial}{\partial t}\right)e_{\alpha\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}91)$$

where $\Delta = \mathrm{U}_{mn}e_{mn}$ is the dilatation.

From the condition of stability of the standard configuration (1·841) we obtain the well-known conditions

$$\lambda + \tfrac{2}{3}\mu \geqq 0, \quad \mu \geqq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}92)$$

for the Lamé elastic constants, λ and μ.

From the fact that the dissipation function F $(= a_{\alpha\beta mn}\dot{e}_{\alpha\beta}\dot{e}_{mn})$ must be positive, we obtain the conditions

$$\lambda' + \tfrac{2}{3}\mu' \geqq 0, \quad \mu' \geqq 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}93)$$

between the corresponding viscous coefficients, λ' and μ'.

§ 2. The Vibrations of a Visco-Elastic Solid: Some General Theorems.

2·1. *The general mathematical problem.*

When a solid body is in a certain state of strain (displacement) and straining (motion), and it is then acted upon by certain constraints applied to its surface, it seems to be almost self evident that the resulting motion of the body is completely and uniquely determined. We now investigate the equivalent mathematical problem, namely, the solution of the equations of motion of a visco-elastic solid subject to given surface (boundary) conditions and given initial conditions of displacement and velocity.

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* See, for example, Jeffreys, "Cartesian Tensors," p. 70.

† $\mathrm{U}_{\alpha\beta}$ is a component of the second order substitution tensor.

We take the stress-strain relations in the form (1·821), namely,

$$p_{\beta\alpha} = p_{\alpha\beta} = c_{\alpha\beta mn} e_{mn} + a_{\alpha\beta mn}\, \dot{e}_{mn}. \quad \ldots \ldots \ldots \quad (2\cdot11)$$

The equations of motion (1·72) become, on substituting this value for the stress components,

$$\rho \frac{\partial^2 u_\alpha}{\partial t^2} = \frac{\partial}{\partial x_\beta}(p_{\beta\alpha}) = \frac{\partial}{\partial x_\beta}(c_{\alpha\beta mn} e_{mn} + a_{\alpha\beta nn}\, \dot{e}_{mn}). \quad \ldots \ldots \quad (2\cdot12)$$

Now in this equation, as we have already neglected thermo-elastic effects, the elastic coefficients $c_{\alpha\beta mn}$ are constants; the viscous coefficients $a_{\alpha\beta mn}$, although independent of the temperature, may still be functions of the strain $\mathbf{e}$ (see § 1·6); and the density is also dependent on the strain according to (1·791), *i.e.*,

$$\rho\,(1 + \mathrm{U}_{mn} e_{mn}) = \rho_0.$$

Now the strain components in (2·12) are determined by the gradient of the displacement according to

$$e_{mn} = \tfrac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right).$$

It is supposed that the displacement u_α ($\alpha = 1, 2, 3$) is so small that the density ρ and the viscous coefficients $a_{\alpha\beta mn}$ in (2·12) can be replaced by the constant values they assume when there is no strain. The equations of motion (2·12) are thus linear in the displacement components. In this approximation we have still kept to the same order of approximation that was involved in the expression of the strain-energy function W (*cf.* (1·780)).

Now the α-component of force across the surface at time t, and measured per unit area *at time t*, is given by*

$$\mathrm{P}_\alpha = p_{\nu\alpha} = \nu_\beta p_{\beta\alpha}, \quad \ldots \ldots \ldots \ldots \quad (2\cdot13)$$

where ν denotes the direction of the normal to the surface at time t, and ν_β is the cosine of the angle between the directions denoted by ν and β. As an approximation it is supposed that ν denotes the direction of the normal to the same element of the surface in the standard configuration, and that equation (2·13) gives the α-component of force measured per unit area *in the standard configuration.*† It is readily seen that this approximation only involves the neglect of terms in (2·13) of the second order in the displacement components u_α; so that the approximation still keeps to the same order of approximation as that already assumed in the expression of the strain energy function W.

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* The reader is again reminded that repetition of a dummy suffix indicates summation over all possible values of that suffix.

† For instance, in problems of the vibrations of cylinders it is always assumed that the normal to the curved surface is perpendicular to the axis, though it is not exactly so.

It is not necessary in what follows that the solid should be uniform. We remind the reader of the condition for stability of equilibrium of the standard configuration (1·841), *i.e.*,

$$c_{\alpha\beta mn}\, e_{\alpha\beta}\, e_{mn} > 0\,; \quad \ldots \ldots \ldots \ldots \quad (2\cdot 14)$$

and of the condition that the dissipation function is essentially positive, namely,

$$a_{\alpha\beta mn}\, \dot{e}_{\alpha\beta}\, \dot{e}_{mn} > 0. \quad \ldots \ldots \ldots \ldots \quad (2\cdot 15)$$

2·2. *Uniqueness of solution.*

Theorem I.—*The solution of the equations of motion of a visco-elastic solid, subject to given surface conditions and a given initial displacement and velocity for every point of the solid, is a unique solution.*

For, let there be two such solutions, which we write as

$$u_\alpha = u_\alpha^{(1)} \quad (\alpha = 1, 2, 3),$$

and

$$u_\alpha = u_\alpha^{(2)} \quad (\alpha = 1, 2, 3).$$

The equations of motion (2·12), namely,

$$\rho\, \frac{\partial^2 u_\alpha}{\partial t^2} = \frac{\partial}{\partial x_\beta}\,(p_{\beta\alpha})$$

are linear in the displacement components u_α ($\alpha = 1, 2, 3$).

Thus the displacement represented by

$$u_\alpha = u_\alpha^{(1)} - u_\alpha^{(2)} \quad (\alpha = 1, 2, 3)$$

is a solution of the equations of motions; that is

$$\rho\, \frac{\partial^2 u_\alpha}{\partial t^2} = \frac{\partial}{\partial x_\beta}\,(p_{\beta\alpha}), \quad \ldots \ldots \ldots \ldots \quad (2\cdot 21)$$

where the stress components $p_{\beta\alpha}$ are calculated from the strain due to displacement components $u_\alpha = u_\alpha^{(1)} - u_\alpha^{(2)}$.

From the equation (2·21), we form the integral equation

$$\int_0^t dt \iiint \frac{\partial u_\alpha}{\partial t}\left(\rho\, \frac{\partial^2 u_\alpha}{\partial t^2} - \frac{\partial}{\partial x_\beta}\, p_{\beta\alpha}\right) d\tau = 0, \quad \ldots \ldots \ldots \quad (2\cdot 22)$$

in which the volume integral is taken over the volume in the standard configuration.

Now*

$$\int_0^t dt \iiint \rho\, \frac{\partial u_\alpha}{\partial t}\, \frac{\partial^2 u_\alpha}{\partial t^2}\, d\tau = \iiint \tfrac{1}{2}\rho \left[\sum_\alpha \left(\frac{\partial u_\alpha}{\partial t}\right)^2\right]_0^t d\tau.$$

---

[*] The density ρ is constant here; the viscous coefficients $a_{\alpha\beta mn}$ are also constant. These approximations were introduced in the last section (2·1).

But

$$\frac{\partial u_\alpha}{\partial t} = \frac{\partial u_\alpha^{(1)}}{\partial t} - \frac{\partial u_\alpha^{(2)}}{\partial t}$$

and vanishes when $t = 0$ since $\partial u_\alpha^{(1)}/\partial t$ and $\partial u_\alpha^{(2)}/\partial t$ are each equal to the given initial velocity. Hence

$$\int_0^t dt \iiint \rho \frac{\partial u_\alpha}{\partial t} \frac{\partial^2 u_\alpha}{\partial t^2} d\tau = \iiint \rho \, \mathrm{T} \, d\tau, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2 \cdot 23)$$

where T is the kinetic energy per unit mass at time t, and is therefore essentially positive.

The remainder of the integral in (2·22) can be transformed into a surface integral and a volume integral. For

$$-\int_0^t dt \iiint \frac{\partial u_\alpha}{\partial t} \frac{\partial}{\partial x_\beta} p_{\beta\alpha} \, d\tau = -\int_0^t dt \iiint \left[\frac{\partial}{\partial x_\beta} \left\{ p_{\beta\alpha} \frac{\partial u_\alpha}{\partial t} \right\} - p_{\beta\alpha} \frac{\partial}{\partial x_\beta} \frac{\partial u_\alpha}{\partial t} \right] d\tau. \quad (2 \cdot 24)$$

Using Green's Transformation to transform the integral of the first term, we have

$$-\int_0^t dt \iiint \frac{\partial}{\partial x_\beta} \left\{ p_{\beta\alpha} \frac{\partial u_\alpha}{\partial t} \right\} d\tau = -\int_0^t dt \iint p_{\nu\alpha} \frac{\partial u_\alpha}{\partial t} d\mathrm{S}. \quad . \; . \; . \; . \quad (2 \cdot 25)$$

The integral of the second term in (2·24) can be reduced to simpler form, since

$$\begin{aligned} p_{\beta\alpha} \frac{\partial}{\partial x_\beta} \frac{\partial u_\alpha}{\partial t} &= p_{\beta\alpha} \frac{\partial}{\partial t} \frac{\partial u_\alpha}{\partial x_\beta} \\ &= p_{\beta\alpha} \frac{\partial}{\partial t} (e_{\alpha\beta} - \mathrm{E}_{\alpha\beta\gamma} \varpi_\gamma) \text{ by (1·22) and (1·23).} \end{aligned}$$

But $\mathrm{E}_{\alpha\beta\gamma} p_{\beta\alpha} = 0$ by (1·73), so that

$$p_{\beta\alpha} \frac{\partial}{\partial x_\beta} \frac{\partial u_\alpha}{\partial t} = p_{\beta\alpha} \frac{\partial e_{\alpha\beta}}{\partial t} = c_{\alpha\beta mn} \, e_{mn} \, \dot{e}_{\alpha\beta} + a_{\alpha\beta mn} \, \dot{e}_{mn} \, \dot{e}_{\alpha\beta} \, ;$$

and the integral of the second term in (2·24) is equal to

$$\int_0^t dt \iiint (c_{\alpha\beta mn} \, e_{mn} \, \dot{e}_{\alpha\beta} + a_{\alpha\beta mn} \, \dot{e}_{mn} \, \dot{e}_{\alpha\beta}) \, d\tau.$$

Now $a_{\alpha\beta mn} \, \dot{e}_{mn} \, \dot{e}_{\alpha\beta} = 2\mathrm{F}$, which is essentially positive; and

$$\frac{\partial}{\partial t} (c_{\alpha\beta mn} \, e_{mn} \, e_{\alpha\beta}) = c_{\alpha\beta mn} (\dot{e}_{mn} e_{\alpha\beta} + e_{mn} \, \dot{e}_{\alpha\beta}) = 2 c_{\alpha\beta mn} \, e_{mn} \, \dot{e}_{\alpha\beta} \quad \text{since} \quad c_{\alpha\beta mn} = c_{mn\alpha\beta}.$$

Thus

$$\int_0^t dt \iiint c_{\alpha\beta mn} \, e_{mn} \, \dot{e}_{\alpha\beta} \, d\tau = \tfrac{1}{2} \iiint \Big[c_{\alpha\beta mn} \, e_{mn} \, e_{\alpha\beta} \Big]_0^t d\tau.$$

But $e_{\alpha\beta} = 0$ when $t = 0$, since $u_\alpha^{(1)}$ and $u_\alpha^{(2)}$ are each equal to the given initial component displacement.

Hence

$$\int_0^t dt \iiint p_{\beta\alpha} \frac{\partial}{\partial x_\beta} \frac{\partial u_\alpha}{\partial t}\, d\tau = \tfrac{1}{2} \iiint c_{\alpha\beta mn}\, e_{mn}\, e_{\alpha\beta}\, d\tau + \int_0^t dt \iiint a_{\alpha\beta mn}\, \dot{e}_{mn}\, \dot{e}_{\alpha\beta}\, d\tau. \qquad (2{\cdot}26)$$

Both these integrals are essentially positive, since their integrands are essentially positive—the first on account of the condition for the stability of equilibrium of the standard configuration (2·14), the second on account of the condition that the dissipation function is essentially positive (2·15). We have now expressed the integral in (2·22) as the sum of three volume integrals which are essentially positive (expressions (2·23) and (2·26)), and a surface integral (2·25). We proceed to show that the surface integral vanishes.

The given surface conditions may be of various types. One or the other (or a combination of both) of the two following types of surface conditions is the type of condition that holds in the case of either forced or free vibrations.

(*a*) The surface of the solid is forced to assume a certain shape, possibly varying with the time, by means of forces or constraints applied to the surface. Thus the component displacements u_α are given as a function of t at the surface of the solid.

(*b*) The surface of the solid may be subject to certain given tractions whose components P_α are defined as functions of the time. The case of free vibrations is included in this class, as it is simply the case when the given value of the surface traction is always zero.

A surface condition of type (*a*) may apply to one portion of the surface, and a surface condition of type (*b*) to the remainder; but the two types of condition can not, in general, hold together over the same portion of the surface.

Over the portion of the surface where the surface conditions are of type (*a*), $u_\alpha^{(1)}$ and $u_\alpha^{(2)}$ are identically equal, so that $u_\alpha = u_\alpha^{(1)} - u_\alpha^{(2)}$ vanishes identically. Hence

$$\frac{\partial u_\alpha}{\partial t} = 0.$$

Now by (2·13) $P_\alpha = p_{\nu\alpha}$; so that, over the portion of the surface where the surface conditions are of type (*b*), $p_{\nu\alpha}^{(1)}$ and $p_{\nu\alpha}^{(2)}$ are identically equal. Hence

$$p_{\nu\alpha} = p_{\nu\alpha}^{(1)} - p_{\nu\alpha}^{(2)} = 0.$$

Hence $p_{\nu\alpha} \dfrac{\partial u_\alpha}{\partial t} = 0$ over all the surface.

Hence the surface integral (2·25) vanishes, for

$$-\int_0^t dt \iint p_{\nu\alpha} \frac{\partial u_\alpha}{\partial t}\, dS = 0. \quad \ldots\ldots\ldots\ldots \quad (2{\cdot}27)$$

Thus we obtain finally from the transformation of the original integral equation (2·22)

$$\iiint \rho T\, d\tau + \tfrac{1}{2} \iiint c_{\alpha\beta mn}\, e_{\alpha\beta}\, e_{mn}\, d\tau + \int_0^t dt \iiint a_{\alpha\beta mn}\, \dot{e}_{\alpha\beta}\, \dot{e}_{mn}\, d\tau = 0. \quad . \quad (2{\cdot}28)$$

As each of these three integrals is essentially positive, each integral must vanish separately. Further, as each integrand is essentially positive, each integrand must vanish separately.* It follows that the values of the displacement components u_α must be constant. But initially $u_\alpha = 0$ since $u_\alpha^{(1)}$ and $u_\alpha^{(2)}$ are each equal to the α-component of the initial displacement. Hence

$$u_\alpha^{(1)} - u_\alpha^{(2)} = u_\alpha = 0, \quad (\alpha = 1, 2, 3) \text{ always.}$$

The two solutions

$$u_\alpha = u_\alpha^{(1)} \quad (\alpha = 1, 2, 3)$$

and

$$u_\alpha = u_\alpha^{(2)} \quad (\alpha = 1, 2, 3),$$

are thus identically the same; and any solution of the equations of motion, subject to given surface conditions and given initial conditions of displacement and velocity for every point of the solid, is thus a unique solution—which proves *Theorem I.*

2·3. *The problem of the free vibrations of a visco-elastic solid: some general theorems.*

Now, it is well known that the free vibrations of a perfectly elastic solid have properties analogous to those of the vibrations of a conservative mechanical system.† It is to be expected that the free vibrations of a visco-elastic solid will have some of the properties of a dissipative mechanical system—not the more general cyclic system. For, on account of the symmetry relation $a_{\alpha\beta mn} = a_{mn\alpha\beta}$ between the viscous coefficients there are no terms in the equations of motion of a visco-elastic solid analogous to the gyrostatic (or cyclic) terms which occur in the equations of motion of a cyclic system.‡

The mathematical problem is to solve the equations of motion (2·12), namely

$$\rho \frac{\partial^2 u_\alpha}{\partial t^2} = \frac{\partial}{\partial x_\beta}(p_{\beta\alpha}) = \frac{\partial}{\partial x_\beta}\left(c_{\alpha\beta mn}\, e_{mn} + a_{\alpha\beta mn} \frac{\partial e_{mn}}{\partial t}\right), \quad . \quad . \quad . \quad . \quad (2{\cdot}311)$$

subject to the condition that there is no traction across the surface, which is

$$p_{\nu\alpha} = \nu_\beta\, p_{\beta\alpha} = \nu_\beta \left(c_{\alpha\beta mn}\, e_{mn} + a_{\alpha\beta mn} \frac{\partial}{\partial t}\, e_{mn}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}312)$$

using the expression (2·13) for the surface traction.

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* In the case of initial stress the condition for the stability of equilibrium of the standard configuration is (1·840), so that the second volume integral is still positive though its integrand is not essentially positive. Results follow in the same way as in the case of no initial stress. This is also true for other theorems in § 2.

† See Lord RAYLEIGH'S "Theory of Sound," vol. 1, and LOVE'S "Elasticity," p. 179.

‡ See, for instance, LAMB'S "Higher Mechanics," p. 234.

There are three equations of motion and three surface conditions according as α indicates the directions 1, 2, or 3 of the rectangular axes; and the suffixes β, m, n are dummy suffixes. The density ρ and the viscous coefficients $a_{\alpha\beta mn}$, as well as the elastic coefficients $c_{\alpha\beta mn}$, are taken as constants according to the approximations discussed in § 2·1.

A discussion of the existence of a solution of these equations does not seem possible; but, on the assumption that a solution of a certain type exists, it is possible to discuss some of the properties of the vibratory motion it represents. We look for a solution of the type

$$\mathbf{u} = \boldsymbol{\phi}\,(x_1, x_2, x_3)\, e^{pt},$$

where $\boldsymbol{\phi}$ is a function of the co-ordinates x_1, x_2, x_3 only, and e^{pt} is written for exp. (pt).

This vector solution is equivalent to a set of solutions for the component displacements, such as

$$u^{\alpha} = \phi_{\alpha}\, e^{pt} \quad (\alpha = 1, 2, 3). \qquad (2{\cdot}321)$$

Substituting this set of solutions for the component displacements into the equations of motion (2·311), we have

$$\rho p^2 \phi_{\alpha} = \frac{\partial}{\partial x_{\beta}} (c_{\alpha\beta mn} + p a_{\alpha\beta mn})\, e_{mn}. \qquad (2{\cdot}322)$$

Here the strain components e_{mn} *are calculated from the displacement represented by* $u_{\alpha} = \phi_{\alpha}$ $(\alpha = 1, 2, 3)$.

These equations are three partial differential equations of the second degree and hold throughout the volume of the solid. The three independent variables are x_{α} $(\alpha = 1, 2, 3)$; and the three dependent variables are ϕ_{α} $(\alpha = 1, 2, 3)$.

The surface conditions (2·312) become, on substitution of the values of the displacement given by the solution (2·321),

$$\nu_{\beta}\, (c_{\alpha\beta mn} + p a_{\alpha\beta mn})\, e_{mn} = 0. \qquad (2{\cdot}323)$$

Here the strain components are defined as in (2·322) above, *i.e.*, $e_{mn} = \frac{1}{2}\left(\frac{\partial \phi_m}{\partial x_n} + \frac{\partial \phi_n}{\partial x_m}\right)$. These are three surface conditions holding over the whole surface of the solid.

It seems that the differential equations (2·322) will provide a solution for the displacement components in terms of p, say

$$\phi_{\alpha} = \chi_{\alpha}\,(x_1, x_2, x_3, p), \; (\alpha = 1, 2, 3). \qquad (2{\cdot}324)$$

This solution will contain arbitrary constants, in addition to the arbitrary multiplying constant, which it obviously contains. Substituting these values for ϕ_{α} in the surface conditions—where the co-ordinates x_{α} have definite values—it is found in soluble problems that the values of the arbitrary constants in the solution (2·324) can be determined (except for the multiplying constant which is clearly still arbitrary) and that an equation for p can be obtained. This equation is called the *frequency equation.*

To any root p of the frequency equation, there correspond a set of functions ϕ_α, determined uniquely by the expressions (2·324) for ϕ_α, except for the arbitrary multiplying constant.

Now, in the case of perfect elasticity, $a_{\alpha\beta mn} = 0$; and p occurs only as p^2 in the equations of motion, and is absent from the surface conditions. Thus it can occur in the frequency equation only as p^2. It is found, in soluble problems in the case of a perfectly elastic solid, that the frequency equation is a transcendental equation for p^2; so that there are an enumerably infinite number of pairs of equal and opposite roots for p.

In the more complicated case of visco-elasticity it seems reasonable to suppose, in the first place, that the frequency equation does not contain any imaginary terms explicitly, on account of the real form of the differential equations (2·322) and the surface conditions (2·323). We assume further that the frequency equation is transcendental, and that p occurs in the frequency equation in quadratic form. There will thus be an enumerably infinite set of pairs of roots for p, which we denote by $p_r^{(1)}$, $p_r^{(2)}$ (r being a positive integer). A particular pair of roots $p_r^{(1)}$, $p_r^{(2)}$ must be both real, or complex conjugates.

Also, in view of the real form of the differential equations (2·322), it is reasonable to suppose that the solutions (2·324) for ϕ_α in terms of p do not contain any imaginary terms explicitly. Thus the set of functions ϕ_α corresponding to a real root of the frequency equation will be real; and the two sets of functions corresponding to a pair of complex conjugate roots of the frequency equation will be complex conjugate sets of functions. We denote the sets of functions ϕ_α corresponding to a pair of roots $p_r^{(1)}$, $p_r^{(2)}$ of the frequency equation by $\phi_\alpha^{(r)(1)}$ and $\phi_\alpha^{(r)(2)}$ ($\alpha = 1, 2, 3$) respectively.

The equations (2·324), from which the functions ϕ_α are calculated for any value of p which is a root of the frequency equation, contain an arbitrary multiplying constant —the value of the constant is the same for each value of α, of course. It is supposed that the value of the arbitrary constant is the same in the calculation of $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ which correspond to a *pair* of roots of the frequency equation.

The sets of functions $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ are determined uniquely by imposing the normalising relation

$$\iiint \rho\, \phi_\alpha^{(r)(1)} \phi_\alpha^{(r)(2)}\, d\tau = 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2\cdot331)$$

the volume integral extending over the volume of the solid.

This is taken as the normalising relation irrespective of whether $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ are real or complex. If the roots $p_r^{(1)}$, $p_r^{(2)}$ are complex conjugates then $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ must be of the form

$$\phi_\alpha^{(r)(1)} = \phi_\alpha^{(r)'} + i\phi_\alpha^{(r)''}, \quad \phi_\alpha^{(r)(2)} = \phi_\alpha^{(r)'} - i\phi_\alpha^{(r)''},$$

where $\phi_\alpha^{(r)'}$, $\phi_\alpha^{(r)''}$ are both real functions. If the roots $p_r^{(1)}$, $p_r^{(2)}$ are both real, then the corresponding sets of functions $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ must be both real, or both entirely imaginary—the arbitrary constant determined by the normalising relation (2·331) may be real or imaginary in this case, but not complex.

There are, therefore, an enumerably infinite set of pairs of solutions of the equations of motion, subject to the condition of no traction across the surface, of the form

$$u_\alpha = \mathrm{L}_r \phi_\alpha^{(r)(1)} e^{p_r^{(1)}t}\,; \quad u_\alpha = \mathrm{M}_r \phi_\alpha^{(r)(2)} e^{p_r^{(2)}t}, \quad (\alpha = 1, 2, 3), \quad . \; . \quad (2\cdot332)$$

where L_r, M_r are arbitrary constants.*

The character of the vibratory motion represented by the solutions (2·332) for the displacement components depends on the values of $p_r^{(1)}$, $p_r^{(2)}$. We prove the following theorem.

THEOREM II.—*Any " pair " of roots of the frequency equation for the free vibration of a visco-elastic solid are roots of a quadratic equation with real positive coefficients.*

We suppose that two solutions (not necessarily a pair) for the displacement components u_α have been obtained in the manner described above ; say

$$u_\alpha = u_\alpha^{(k)} = \phi_\alpha^{(k)} e^{p_k t} \quad (\alpha = 1, 2, 3), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2\cdot341)$$

and

$$u_\alpha = u_\alpha^{(l)} = \phi_\alpha^{(l)} e^{p_l t} \quad (\alpha = 1, 2, 3). \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2\cdot342)$$

The sets of functions $\phi_\alpha^{(k)}$ and $\phi_\alpha^{(l)}$ have been normalised according to (2·331) and are thus uniquely determined.

Since (2·341) is a solution, from (2·322) we obtain

$$\rho p_k^2 \phi_\alpha^{(k)} = \frac{\partial}{\partial x_\beta}\left(c_{\alpha\beta mn} + p_k a_{\alpha\beta mn}\right) e_{mn}^{(k)}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2\cdot343)$$

where the strain components $e_{mn}^{(k)}$ are *calculated from the displacement components* $u_\alpha = \phi_\alpha^{(k)}$, $(\alpha = 1, 2, 3)$.

From (2·343) we form the integral equation

$$\iiint \left[\rho p_k^2 \phi_\alpha^{(k)} - \frac{\partial}{\partial x_\beta}\left(c_{\alpha\beta mn} + p_k a_{\alpha\beta mn}\right) e_{mn}^{(k)}\right] \phi_\alpha^{(l)}\, d\tau = 0, \quad . \; . \; . \quad (2\cdot344)$$

where the volume integral extends over the volume of the solid.

Now

$$\iiint \frac{\partial}{\partial x_\beta}\left[\left(c_{\alpha\beta mn} + p_k a_{\alpha\beta mn}\right) e_{mn}^{(k)}\right] \phi_\alpha^{(l)}\, d\tau$$

$$= \iiint \left\{\frac{\partial}{\partial x_\beta}\left[\left(c_{\alpha\beta mn} + p_k a_{\alpha\beta mn}\right) e_{mn}^{(k)}\, \phi_\alpha^{(l)}\right] - \left[\left(c_{\alpha\beta mn} + p_k a_{\alpha\beta mn}\right) e_{mn}^{(k)}\right] \frac{\partial}{\partial x_\beta}\, \phi_\alpha^{(l)}\right\} d\tau$$

$$= \iint \nu_\beta \left(c_{\alpha\beta mn} + p_k a_{\alpha\beta mn}\right) e_{mn}^{(k)} \phi_\alpha^{(l)}\, d\mathrm{S} - \iiint \left(c_{\alpha\beta mn} + p_k a_{\alpha\beta mn}\right) e_{mn}^{(k)} \frac{\partial}{\partial x_\beta}\, \phi_\alpha^{(l)}\, d\tau,$$

using GREEN's Transformation for the first term in the volume integral.

---

* The summation convention is not taken to hold for the repeated suffix r, only for a repeated suffix which indicates an axial direction.

The surface integral vanishes by (2·323). Now

$$\frac{\partial}{\partial x_\beta}\phi_\alpha^{\,l} = e_{\alpha\beta}^{(l)} - \mathrm{E}_{\alpha\beta\gamma}\varpi_\gamma^{(l)}$$

from (1·22) and (1·23). Also

$$\mathrm{E}_{\alpha\beta\gamma}c_{\alpha\beta mn} = 0 \quad \text{and} \quad \mathrm{E}_{\alpha\beta\gamma}a_{\alpha\beta mn} = 0$$

since

$$c_{\alpha\beta mn} = c_{\beta\alpha mn} \quad \text{and} \quad a_{\alpha\beta mn} = a_{\beta\alpha mn}.$$

Thus the integral equation (2·344) becomes

$$\iiint \left[\rho p_k^2 \phi_\alpha^{(k)}\phi_\alpha^{(l)} + (c_{\alpha\beta mn} + p_k a_{\alpha\beta mn})\, e_{mn}^{(k)}\, e_{\alpha\beta}^{(l)}\right] d\tau = 0.$$

We write this as

$$p_k^2\mathrm{A}\,(k,\,l) + p_k\mathrm{B}\,(k,\,l) + \mathrm{C}\,(k,\,l) = 0, \quad \ldots\ldots \quad (2{\cdot}345)$$

where

$$\left.\begin{aligned} \mathrm{A}\,(k,\,l) &= \iiint \rho\,\phi_\alpha^{(k)}\phi_\alpha^{(l)}\,d\tau\,; \\ \mathrm{B}\,(k,\,l) &= \iiint a_{\alpha\beta mn}e_{mn}^{(k)}e_{\alpha\beta}^{(l)}d\tau\,; \\ \mathrm{C}\,(k,\,l) &= \iiint c_{\alpha\beta mn}e_{mn}^{(k)}e_{\alpha\beta}^{(l)}d\tau. \end{aligned}\right\} \quad \ldots\ldots \quad (2{\cdot}346)$$

Now

$$a_{\alpha\beta mn}e_{mn}^{(k)}e_{\alpha\beta}^{(l)} = a_{\alpha\beta mn}e_{mn}^{(l)}e_{\alpha\beta}^{(k)},$$

since

$$a_{\alpha\beta mn} = a_{mn\alpha\beta}.$$

Hence

$$\mathrm{B}\,(k,\,l) = \mathrm{B}\,(l,\,k)\,;$$

and similarly

$$\mathrm{C}\,(k,\,l) = \mathrm{C}\,(l,\,k).$$

Also

$$\mathrm{A}\,(k,\,l) = \mathrm{A}\,(l,\,k).$$

The result (2·345) could clearly have been obtained with the superscripts k and l interchanged; so that

$$p_l^2\,\mathrm{A}\,(l,\,k) + p_l\,\mathrm{B}\,(l,\,k) + \mathrm{C}\,(l,\,k) = 0.$$

Hence, since A (k, l), B (k, l) and C (k, l) are symmetrical in k and l, it follows that p_k and p_l are the roots of the equation*

$$p^2\,\mathrm{A}\,(k,\,l) + p\,\mathrm{B}\,(k,\,l) + \mathrm{C}\,(k,\,l) = 0 \quad \ldots\ldots \quad (2{\cdot}347)$$

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[*] The similarity of equation (2·347) to a similar quadratic equation in the theory of a dissipative mechanical system should be noticed. The equation is $\lambda^2\mathrm{T}\,(\alpha,\,\alpha') + \lambda\mathrm{F}\,(\alpha,\,\alpha') + \mathrm{V}\,(\alpha,\,\alpha') = 0$ (see LAMB's ‘ Higher Mechanics,” p. 232, equation (8)). Here λ corresponds to our p. The analogy between the function B and the dissipative function F in a mechanical system, and between the function C and the potential energy function V in a mechanical system should be noticed.

The result (2·345) could have been obtained in just the same way if k and l are equal. Thus p_k is a root of

$$p^2 \mathrm{A}(k, k) + p \mathrm{B}(k, k) + \mathrm{C}(k, k) = 0,$$

though not necessarily a double root.

Let the superscripts (k) and (l) now be the superscripts (r) (1) and (r) (2) of a *pair* of solutions. Thus, from (2·347) we see that a *pair* of roots $p_r^{(1)}$, $p_r^{(2)}$ of the frequency equation are roots of the quadratic equation

$$\mathrm{A}_r p^2 + \mathrm{B}_r p + \mathrm{C}_r = 0, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2\cdot348)$$

where A_r has been written for $\mathrm{A}(r1, r2)$; B_r for $\mathrm{B}(r1, r2)$; and C_r for $\mathrm{C}(r1, r2)$.

To prove theorem II, it remains to show that A_r, B_r and C_r are always positive. Now A_r is positive because

$$\mathrm{A}_r = \iiint \rho \phi_\alpha^{(r)(1)} \phi_\alpha^{(r)(2)} \, d\tau = 1,$$

by the normalising relation (2·331).

To show that B_r and C_r are positive it is necessary to consider separately the case in which $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ are complex conjugates, and the case in which they are both real (or wholly imaginary).

We consider first the case in which $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ are complex; and given by

$$\phi_\alpha^{(r)(1)} = \phi_\alpha^{(r)'} + i\phi_\alpha^{(r)''}, \quad \phi_\alpha^{(r)(2)} = \phi_\alpha^{(r)'} - i\,\phi_\alpha^{(r)''}, \quad \text{say},$$

where $\phi_\alpha^{(r)'}$, $\phi_\alpha^{(r)''}$ are both entirely real.

Thus

$$e_{mn}^{(r)(1)} = e_{mn}^{(r)'} + ie_{mn}^{(r)''} \quad \text{and} \quad e_{mn}^{(r)(2)} = e_{mn}^{(r)'} - ie_{mn}^{(r)''};$$

so that

$$\begin{aligned} \mathrm{B}_r &= \iiint a_{\alpha\beta mn} (e_{mn}^{(r)'} + ie_{mn}^{(r)''}) (e_{\alpha\beta}^{(r)'} - ie_{\alpha\beta}^{(r)''}) \, d\tau \\ &= \{\mathrm{B}(r', r') + \mathrm{B}(r'', r'')\} + i\{\mathrm{B}(r'', r') - \mathrm{B}(r', r'')\} \\ &= \mathrm{B}(r', r') + \mathrm{B}(r'', r''), \end{aligned}$$

since $\mathrm{B}(r'', r') = \mathrm{B}(r', r'')$ by the symmetry property of $\mathrm{B}(k, l)$.

Now the condition that the dissipation for the visco-elastic solid is essentially positive is given by the condition (2·15), namely,

$$a_{\alpha\beta mn} \dot{e}_{mn} \dot{e}_{\alpha\beta} > 0.$$

Let the rate of strain components in this expression be calculated from the displacement represented by

$$u_\alpha = \phi_\alpha^{(r)'} t \quad (\alpha = 1, 2, 3).$$

The rate of strain components are thus $e_{mn}^{(r)'}$, and the condition that the dissipation function is essentially positive becomes

$$a_{\alpha\beta mn} e_{mn}^{(r)'} e_{\alpha\beta}^{(r)'} > 0.$$

Hence

$$\mathrm{B}\,(r', r') = \iiint a_{\alpha\beta mn} e_{mn}^{(r)'} e_{\alpha\beta}^{(r)'}\, d\tau > 0.$$

Similarly

$$\mathrm{B}\,(r'', r'') > 0\,;$$

so that we have

$$\mathrm{B}_r > 0.$$

In the same way we show that

$$\mathrm{C}_r = \mathrm{C}\,(r', r') + \mathrm{C}\,(r'', r'')\,;$$

and each of these terms is positive on account of the condition (2·14) for the stability of equilibrium of the standard configuration. Hence $\mathrm{C}_r > 0$.

We now consider the case when $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ are both wholly real or both wholly imaginary. Now $p_r^{(1)}$ is a root of the equation

$$\mathrm{A}\,(r1, r1)\, p^2 + \mathrm{B}\,(r1, r1)\, p + \mathrm{C}\,(r1, r1) = 0.$$

If the $\phi_\alpha^{(r)(1)}$ are real, then

$$\mathrm{A}\,(r1, r1) = \iiint \rho\, \phi_\alpha^{(r)(1)} \phi_\alpha^{(r)(1)}\, d\tau > 0,$$

since the integrand is the sum of squares.

Also we can show that $\mathrm{B}\,(r1, r1) > 0$, in exactly the same way that it was shown that $\mathrm{B}\,(r', r') > 0$, this result being due to the essentially positive character of the dissipation function.

Similarly, we show that $\mathrm{C}\,(r1, r1) > 0$, from the condition for the stability of equilibrium of the standard configuration.

Thus $\mathrm{A}\,(r1, r1)$, $\mathrm{B}\,(r1, r1)$, $\mathrm{C}\,(r1, r1)$ are all positive.

In the case when the $\phi_\alpha^{(r)(1)}$ are wholly imaginary, we can take the imaginary factors i outside the integral sign, and it immediately follows from the foregoing results that $\mathrm{A}\,(r1, r1)$, $\mathrm{B}\,(r1, r1)$, $\mathrm{C}\,(r1, r1)$ are all negative.

In either case $p_r^{(1)}$ is a root of a quadratic equation with real positive coefficients. Since, by hypothesis, $p_r^{(1)}$ is real in the case under consideration, it follows that it must also be negative. Similarly, we show that $p_r^{(2)}$ is real and negative.

But $p_r^{(1)}$, $p_r^{(2)}$ are roots of equation (2·348), namely,

$$\mathrm{A}_r p^2 + \mathrm{B}_r p + \mathrm{C}_r = 0,$$

in which $\mathrm{A}_r = 1$. Since $p_r^{(1)}$, $p_r^{(2)}$ are in this case real and negative, B_r and C_r are positive.

Thus, whether $p_r^{(1)}$, $p_r^{(2)}$ are real or complex, A_r, B_r, C_r are all positive, which proves *Theorem II.*

We are now in a position to determine the character of the free vibratory motion of a visco-elastic solid. To each pair of roots of the frequency equation there corresponds a pair of solutions for the displacement components given by (2·332). From these solutions we build up the more general solution

$$u_\alpha = \sum_{r=1}^{\infty} \{L_r \phi_\alpha^{(r)(1)} e^{p_r^{(1)}t} + M_r \phi_\alpha^{(r)(2)} e^{p_r^{(2)}t}\}, \quad (\alpha = 1, 2, 3). \quad . \quad . \quad . \quad (2·351)$$

The motion represented by an individual term of this series expression differs according to whether $p_r^{(1)}$, $p_r^{(2)}$ are complex or real. We consider the two cases separately.

(*a*) The pair of roots $p_r^{(1)}$, $p_r^{(2)}$ of the frequency equation are complex. From Theorem II above, it immediately follows that the real part of $p_r^{(1)}$, $p_r^{(2)}$ is negative, so that

$$p_r^{(1)} = -p'_r + ip''_r, \quad p_r^{(2)} = -p'_r - ip''_r,$$

where p'_r, p''_r are positive.

The functions $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ are of the form

$$\phi_\alpha^{(r)(1)} = \phi_\alpha^{(r)'} + i\phi_\alpha^{(2)''}, \quad \phi_\alpha^{(r)(2)} = \phi_\alpha^{(r)'} - i\phi_\alpha^{(r)''}.$$

In order that the motion represented by

$$u_\alpha = L_r \phi_\alpha^{(r)(1)} e^{p_r^{(1)}t} + M_r \phi_\alpha^{(r)(2)} e^{p_r^{(2)}t}, \quad (\alpha = 1, 2, 3)$$

shall be real, L_r and M_r must be complex conjugate constants. The expression for the displacement components thus reduces to the form

$$u_\alpha = A_r e^{-p'_r t} [\phi_\alpha^{(r)'} \cos (p''_r t + \varepsilon_r) - \phi_\alpha^{(r)''} \sin (p''_r{}^t + \varepsilon_r)], \quad (\alpha = 1, 2, 3). \quad .(2·352)$$

This represents a damped harmonic oscillation of arbitrary amplitude and phase in which the period and the damping factor are the same for every point of the solid ; but the phase ε, given by

$$\tan \varepsilon = \frac{\phi_\alpha^{(r)'} \sin \varepsilon_r + \phi_\alpha^{(r)''} \cos \varepsilon_r}{\phi_\alpha^{(r)'} \cos \varepsilon_r - \phi_\alpha^{(r)''} \sin \varepsilon_r},$$

varies from point to point of the solid. This is similar to the damped harmonic oscillation of a dissipative mechanical system, the phase being different, in general, for each particle of such a system.

(*b*) When $p_r^{(1)}$, $p_r^{(2)}$ are real, it follows from Theorem II that they must be negative. We have also seen that the sets of functions $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ are both wholly real or both wholly imaginary in this case. The constants L_r, M_r are thus wholly real or wholly imaginary, but independent of each other, in order that the motion represented by

$$u_\alpha = L_r \phi_\alpha^{(r)(1)} e^{p_r^{(1)} t} + M_r \phi_\alpha^{(r)(2)} e^{p_r^{(2)} t}, \quad (\alpha = 1, 2, 3),$$

shall be real. The motion represented by this expression for the displacement components is the superposition of two exponentially damped aperiodic motions, each of arbitrary initial amplitude.

We have not yet fixed the order of the terms in the general solution (2·351) which is indicated by the suffix r (r is a positive integer).

Now $p_r^{(1)}$, $p^{(2)}$ are roots of

$$A_r p^2 + B_r p + C_r = 0,$$

where $A_r = 1$. B_r and C_r are positive; and are given by the formulæ

$$B_r = \iiint a_{\alpha\beta mn} e_{mn}^{(r)(1)} e_{\alpha\beta}^{(r)(2)} \, d\tau \; ; \quad C_r = \iiint c_{\alpha\beta mn} e_{mn}^{(r)(1)} e_{\alpha\beta}^{(r)(2)} \, d\tau. \quad . \quad (2\cdot353)$$

Thus

$$p_r^{(1)}, p_r^{(2)} = \frac{B_r}{2}\left\{-1 \pm \left[1 - \frac{4C_r}{B_r^2}\right]^{\frac{1}{2}}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2\cdot354)$$

and $p_r^{(1)}$, $p_r^{(2)}$ will be complex or real according as

$$D_r < \text{or} > 1\,;$$

where

$$D_r = B_r^2/4C_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2\cdot355)$$

The D_r are *numbers*—it is readily seen that D_r is dimensionless, bearing in mind the normalising relation (2·331).

It is convenient to take the increasing order of the integers r to coincide with the increasing order of magnitude of the numbers D_r. Thus the complex roots come first in the order, that is, if there are any complex roots.

We have shown that the motion of any point of a visco-elastic solid during its free vibration can be represented as the superposition of an enumerably infinite individual motions of two possible types: (*a*) a damped harmonic oscillation and (*b*) the superposition of two aperiodic motions. The individual motions are of type (*a*) or (*b*), according as $D_r <$ or > 1.

Three possibilities arise.

(1) All the D_r are less than one. The vibration is then the superposition of an enumerably infinite number of damped harmonic oscillations of different frequency.

(2) Some of the D_r are less than one, and some greater than one. The vibration is the superposition of damped harmonic oscillations and aperiodic motions.

(3) All the D_r are greater than one. The vibration is the superposition of an enumerably infinite number of pairs of aperiodic motions, and is thus itself entirely aperiodic. However, the solid is displaced from its equilibrium position and, however it is set in motion it will not oscillate about its equilibrium position,* but the displacement will die away asymptotically with the increase of time, the solid gradually sinking back into its equilibrium position.

The criterion as to whether or not oscillatory motion is possible during the free vibration of the solid depends on whether any D_r are less than one.

We define the " critical number " of a visco-elastic solid as D_1, *the smallest of the* D_r where D_r is given by the formulæ (2·355) and (2·353). We have thus proved the following theorem.

THEOREM III.—*If the " critical number " of a visco-elastic solid is greater than one, then no oscillatory motion is possible during its free vibration.*

By its definition, the value of the critical number for a solid depends on the values of the elastic coefficients $c_{\alpha\beta mn}$, the viscous coefficients $a_{\alpha\beta mn}$, and the solutions $\phi_\alpha^{(1)(1)}$ and $\phi_\alpha^{(1)(2)}$ $(\alpha = 1, 2, 3)$ of the differential equations (2·322) subject to the surface conditions (2·323). These solutions will depend, in general, on the relative dimensions (shape) and the absolute dimensions (size) of the solid, as well as on the elastic and viscous coefficients. Thus the value of the critical number of a visco-elastic solid depends on its shape and size and on the values of the elastic and viscous coefficients. It is of interest to determine the manner in which the value of the critical number depends on these various factors.

We examine only the simple case when the solutions $\phi_\alpha^{(1)(1)}$, $\phi_\alpha^{(1)(2)}$ $(\alpha = 1, 2, 3)$ do not depend on the values of the elastic and viscous coefficients—this is true for certain simple shapes of solids, *cf.*, the solutions of the problems of the vibrations of thin cylinders in § 3.

If the viscous coefficients $a_{\alpha\beta mn}$ are increased in the ratio $k:1$, then B_1 becomes kB_1 while C_1 remains the same. Thus the critical number $D_1 = B_1^2/4C_1$ becomes k^2D_1, being increased in the ratio of $k^2 : 1$. Thus for solids of given shape and size and given elastic coefficients, the less viscous solids are capable of oscillation in their free vibration, while the more viscous ones are not capable of oscillation.

The effect of change of size on the critical number of a visco-elastic solid becomes apparent by the use of a simple transformation. Let the linear dimensions of the solid be increased in the ratio $l:1$, so that the co-ordinates of the larger solid are expressed in terms of the co-ordinates of the smaller solid by the formulæ

$$x'_\alpha = lx_\alpha \quad (\alpha = 1, 2, 3).$$

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* The word " oscillate " is used to describe a backwards and forwards motion. The word " vibration " is used to describe any motion about an equilibrium position.

Thus, $\frac{\partial}{\partial x'_\alpha} \equiv \frac{1}{l} \frac{\partial}{\partial x_\alpha}$ and $e'_{mn} = \frac{1}{l} e_{mn}$; so that the differential equations (2·322) and the surface conditions (2·323) for the larger solid (co-ordinates x'_α) can be written as

$$p^2 \phi_\alpha = \frac{\partial}{\partial x_\beta} \left[\frac{c_{\alpha\beta mn}}{l^2} + p \frac{a_{\alpha\beta mn}}{l^2} \right] e_{mn}$$

and

$$\nu_\beta \left[\frac{c_{\alpha\beta mn}}{l^2} + p \frac{a_{\alpha\beta mn}}{l^2} \right] e_{mn} = 0,$$

where the surface conditions now hold over the surface of the smaller solid. Thus the critical number of a solid, whose linear dimensions are increased in the ratio $l : 1$, is the same as the critical number of the solid whose elastic and viscous coefficients are $c_{\alpha\beta mn}/l^2$ and $a_{\alpha\beta mn}/l^2$. Therefore, B_1 becomes B_1/l^2 and C_1 becomes C_1/l^2; and the critical number becomes D_1/l^2, being decreased in the ratio $1 : l^2$.

Suppose that the critical number of the given solid is D_1, greater than one, say. As the size of the solid is gradually increased, the ratio $l : 1$ of the linear dimensions to the linear dimensions of the given solid gradually increases, starting from the value unity; and the critical number gradually decreases, passing through the value unity when $l = \sqrt{D_1}$.

We call the size of a solid, such that the critical number is unity, the *critical size* of a solid.

When the size is greater than the critical size,

$$l > D_1$$

so that

$$D'_1 = D_1/l^2 < 1.$$

When the size is less than the critical size,

$$l < D_1 \text{ so that } D'_1 = D_1/l^2 > 1.$$

We thus have the following theorem for visco-elastic solids, whose shape is such that the solutions $\phi_\alpha^{(1)(1)}$, $\phi_\alpha^{(1)(2)}$ of the differential equations (2·322) subject to the surface conditions (2·323) are independent of the elastic and viscous coefficients.

THEOREM IV.—*A visco-elastic solid, of given material and shape, is incapable of oscillatory motion in its free vibration if its size is less than the critical size.*

In view of the result that the critical number is increased in the ratio $k^2 : 1$ when the viscous coefficients are increased in the ratio $k : 1$, we have the following corollary.

THEOREM IV. *Cor.* 1.—*For solids of given shape and elastic properties, the critical size is greatest for the most viscous solids.*

For a perfectly elastic solid $a_{\alpha\beta mn} = 0$, so that $D_1 = 0$; and we have the following corollary.

THEOREM IV. *Cor.* 2.—*The critical size of a perfectly elastic solid is vanishingly small.*

When the size of the solid is less than the critical size, the motion represented by (2·351) is the elastic afterworking according to the theory of visco-elasticity. Now, by (2·354),

$$p_r^{(1)},\ p_r^{(2)}=\frac{\mathrm{B}_r}{2}\left\{-1\pm\left[1-\frac{1}{\mathrm{D}_r}\right]^{\frac{1}{2}}\right\},\quad \text{where}\quad \mathrm{D}_r=\mathrm{B}_r^2/4\mathrm{C}_r.$$

When $\mathrm{D}_r \gg 1$, we have approximately

$$p_r^{(1)},\ p_r^{(2)}=\frac{\mathrm{B}_r}{2}\left\{-1\pm\left[1-\frac{1}{2\mathrm{D}_r}\right]\right\}.$$

Thus

$$p_r^{(1)}=-\frac{\mathrm{B}_r}{4\mathrm{D}_r}=-\frac{\mathrm{C}_r}{\mathrm{B}_r},\quad p_r^{(2)}=-\mathrm{B}_r,\quad \text{approximately.}$$

Now $\mathrm{B}_r \gg \frac{\mathrm{C}_r}{\mathrm{B}_r}$ since $\mathrm{D}_r \gg 1$; and the numerical value of $p_r^{(2)}$ is much greater than the numerical value of $p_r^{(1)}$. The motion represented by the least heavily damped terms survives the longest, that is the motion represented by the $p_r^{(1)}$ terms. We note the similarity of the expression for the elastic afterworking in a visco-elastic solid to the empirical formula for the elastic afterworking obtained by NEESEN, $x=ce^{-\beta t}+c_1e^{-\beta_1 t}$ (see the introduction, p. 342).

When the size of the solid is greater than the critical size the motion represented by (2·351) is the superposition of damped harmonic oscillations and elastic afterworking effects (aperiodic motion). The motion represented by the least heavily damped terms again survives the longest, of whichever type these terms may be. It is not evident, in general, whether an aperiodic term or oscillatory term has the least damping factor.

To complete the analytical solution of the problem it is necessary to make the general solution (2·351), namely

$$u_\alpha=\sum_{r=1}^{\infty}\{\mathrm{L}_r\phi_\alpha^{(r)(1)}e^{p_r^{(1)}t}+\mathrm{M}_r\phi_\alpha^{(r)(2)}e^{p_r^{(2)}t}\},\quad (\alpha=1,2,3)$$

satisfy given initial conditions of displacement and velocity for every point of the solid by a suitable choice of the constants L_r and M_r. The initial conditions are

$$(u_\alpha)_{t=0}=\sum_{r=1}^{\infty}\{\mathrm{L}_r\phi_\alpha^{(r)(1)}+\mathrm{M}_r\phi_\alpha^{(r)(2)}\},\quad (\alpha=1,2,3),\quad .\ .\ .\ .\ .\quad (2{\cdot}361)$$

and

$$\left(\frac{\partial u_\alpha}{\partial t}\right)_{t=0}=\sum_{r=1}^{\infty}\left\{\mathrm{L}_r p_r^{(1)}\phi_\alpha^{(r)(1)}+\mathrm{M}_r p_r^{(2)}\phi_\alpha^{(r)(2)}\right\},\quad (\alpha=1,2,3)\quad .\ .\quad (2{\cdot}362)$$

I have been unable to obtain any orthogonal relations between $\phi_\alpha^{(r)(1)}$, $\phi_\alpha^{(r)(2)}$ $(\alpha=1,2,3)$

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and $\phi_{\alpha}^{(s)(1)}$, $\phi_{\alpha}^{(s)(2)}$ ($\alpha = 1, 2, 3$) where $r \neq s$, except in the special case when there is a relation between the elastic and viscous coefficients of the form[*]

$$a_{\alpha\beta mn} = \tau c_{\alpha\beta mn}, \quad (\tau \text{ a constant}).$$

2·4. *Some properties of the free vibrations of a visco-elastic solid for which*[†] $a_{\alpha\beta mn} = \tau c_{\alpha\beta mn}$

If $a_{\alpha\beta mn} = \tau c_{\alpha\beta mn}$, the differential equations (2·322) reduce to

$$\rho \frac{p^2}{1 + \tau p} \phi_{\alpha} = \frac{\partial}{\partial x_{\beta}} c_{\alpha\beta mn} e_{mn} ; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2{\cdot}411)$$

and the surface conditions (2·322) can be written (dividing by $1 + \tau p$) as

$$\nu_{\beta} c_{\alpha\beta mn} e_{mn} = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2{\cdot}412)$$

Now p only occurs as $p^2/[1 + \tau p]$ in the differential equations (2·411), so that the solutions (2·324) for ϕ_{α} in terms of p can contain p only as $p^2/[1 + \tau p]$, *i.e.*,

$$\phi_{\alpha} = \chi_{\alpha}(x_1, x_2, x_3, p^2/[1 + \tau p]), \quad (\alpha = 1, 2, 3). \; . \; . \; . \; . \; . \quad (2{\cdot}413)$$

Indeed, since the surface conditions are the same as for a perfectly elastic solid, the analysis of the problem is the same as in the case of a perfectly elastic solid except so far as $p^2/[1 + \tau p]$ replaces[‡] p^2.

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[*] This is similar to corresponding results in the theory of a dissipative mechanical system. We have already noticed the analogy between B and the dissipation function in a mechanical system, and between C and the potential energy function for a mechanical system (see footnote, p. 377). The particular case when $a_{\alpha\beta mn} = \tau c_{\alpha\beta mn}$ (so that $B = \tau C$) evidently corresponds to the case when the quadratic expressions for the dissipation function and for the potential energy function can be reduced to the sum of squares simultaneously with the expression for the kinetic energy in a mechanical system. Now, unless the dissipation function and the potential energy simultaneously reduce to the sum of squares, in any normal vibration of the system there is a small vibration in $q_2 \, q_3, \; . \; . \; . \; q_n$, besides the vibration in q_1. Thus the initial conditions are, in general, $2n$ linear equations for L_r and M_r ($r = 1, 2, \ldots n$), every L_r and M_r appearing in every equation. But in the case when the dissipation function and the potential energy function simultaneously reduce to the sum of squares, there is a vibration only in q_1 in a typical normal vibration of the system; and the initial conditions reduce to the form

$$(q r)_0 = L_r + M_r \quad \text{and} \quad (\dot{q}_r)_0 = p_r^{(1)} L_r + p_r^{(2)} M_r.$$

These simple initial conditions are analogous to the simple formulæ for calculating L_r, M_r from the initial conditions when there is an orthogonal relation between $\phi_{a}^{(r)(1)}$ and $\phi_{a}^{(s)(1)}$. (For these results concerning a dissipative mechanical system, see, for instance, LAMB'S "Higher Mechanics," p. 233, or WHITTAKER'S "Analytical Dynamics," p. 233.)

[†] The constant τ should not be confused with the element of volume $d\tau$.

[‡] The relation $a_{\alpha\beta mn} = \tau c_{\alpha\beta mn}$ for an æolotropic solid reduces to the relation $\lambda'/\lambda = \mu'/\mu$ for an isotropic solid. We have here a general explanation of why SEZAWA found the problem of the propagation of surface waves in a visco-elastic medium to be soluble, only by making this assumption. See SEZAWA 'Bull. Earthq. Res. Inst. Tokyo Univ.' vol. 3, p. 43 (1927). See the introduction, p. 350.

But (by Theorem II), $p_r^{(1)}$ and $p_r^{(2)}$ are the roots of equation (2·348), namely,

$$A_r p^2 + B_r p + C_r = 0$$

in which $A_r = 1$, and B_r and C_r are positive and real.

Since $a_{\alpha\beta mn} = \tau c_{\alpha\beta mn}$, $B_r = \tau C_r$; and we have

$$\frac{p_r^{(1)2}}{1 + \tau p_r^{(1)}} = -C_r = \frac{p_r^{(2)2}}{1 + \tau p_r^{(2)}} . \quad \text{. . . . .} \quad (2\cdot421)$$

Thus the functions ϕ_α corresponding to the two roots $p_r^{(1)}$ and $p_r^{(2)}$, and given by (2·413), are the same; that is,

$$\phi_\alpha^{(r)(1)} = \phi_\alpha^{(r)} = \phi_\alpha^{(r)(2)} . \quad \text{. . . . .} \quad (2\cdot422)$$

The most general solution for the displacement components, given by (2·351), now becomes

$$u_\alpha = \sum_{r=1}^{\infty} \phi_\alpha^{(r)} \{L_r e^{p_r^{(1)} t} + M_r e^{p_r^{(2)} t}\}, \quad (\alpha = 1, 2, 3). \quad \text{. . . .} \quad (2\cdot431)$$

The initial conditions (2·361) and (2·362) become

$$(u_\alpha)_{t=0} = \sum_{r=1}^{\infty} \phi_\alpha^{(r)} [L_r + M_r], \quad (\alpha = 1, 2, 3). \quad \text{. . . . . .} \quad (2\cdot441)$$

and

$$\left(\frac{\partial u_\alpha}{\partial t}\right)_{t=0} = \sum_{r=1}^{\infty} \phi_\alpha^{(r)} [L_r p_r^{(1)} + M_r p_r^{(2)}], \quad (\alpha = 1, 2, 3). \quad \text{. . .} \quad (2\cdot442)$$

The normalising relation (2·331) becomes simply

$$\iiint \rho \phi_\alpha^{(r)} \phi_\alpha^{(r)} \, d\tau = 1 . \quad \text{. . . . .} \quad (2\cdot443)$$

From (2·347) we see that $p_r^{(1)}$, $p_s^{(1)}$ are the roots of an equation

$$p^2 . A(r, s) + pB(r, s) + C(r, s) = 0 \quad (\text{if } r \neq s),$$

where

$$A(r, s) = \iiint \rho \phi_\alpha^{(r)} \phi_\alpha^{(s)} \, d\tau ; \qquad B(r, s) = \iiint a_{\alpha\beta mn} e_{mn}^{(r)} e_{\alpha\beta}^{(s)} \, d\tau ;$$

$$C(r, s) = \iiint c_{\alpha\beta mn} e_{mn}^{(r)} e_{\alpha\beta}^{(s)} \, d\tau .$$

But the set of functions $\phi_\alpha^{(r)}$ correspond to *both* roots $p_r^{(1)}$ *and* $p_r^{(2)}$. Thus $p_r^{(2)}$ and $p_s^{(1)}$ are also roots of the same equation; and we have

$$[p_r^{(1)} + p_s^{(1)}] A(r, s) = -B(r, s),$$

and

$$[p_r^{(2)} + p_s^{(1)}] A(r, s) = -B(r, s).$$

By subtraction

$$[p_r^{(1)} - p_r^{(2)}]\,\mathrm{A}\,(r, s) = 0.$$

Since $p_r^{(1)} \neq p_r^{(2)}$,

$$\mathrm{A}\,(r, s) = \iiint \rho\,\phi_\alpha^{(r)}\,\phi_\alpha^{(s)}\,d\tau = 0, \quad \text{if } r \neq s. \quad . \ . \ . \ . \ . \quad (2{\cdot}444)$$

If $r = s$, $\mathrm{A}\,(r, s) = 1$, by the normalising relation (2·443).

The constants L_r, M_r can now be determined from the initial conditions (2·441) and (2·442) by the use of the orthogonal property (2·444). Multiplying (2·441) by $\phi_\alpha^{(s)}$ (and summing over all values of α, of course) and integrating throughout the volume of the solid, we obtain

$$\iiint \rho\,(u_\alpha)_{t=0}\,\phi_\alpha^{(s)}\,d\tau = \sum_{r=1}^{\infty}(\mathrm{L}_r + \mathrm{M}_r)\,\mathrm{A}\,(r, s) = \mathrm{L}_s + \mathrm{M}_s. \quad . \ . \quad (2{\cdot}445)$$

Similarly from (2·442) we obtain

$$\iiint \rho\left(\frac{\partial u_\alpha}{\partial t}\right)_{t=0}\phi_\alpha^{(s)}\,d\tau = \sum_{r=1}^{\infty}(p_r^{(1)}\,\mathrm{L}_r + p_r^{(2)}\,\mathrm{M}_r)\,\mathrm{A}\,(r, s) = p_s^{(1)}\,\mathrm{L}_s + p_s^{(2)}\,\mathrm{M}_s. \quad (2{\cdot}446)$$

The constants L_s and M_s (s any positive integer) are immediately calculable.

In this special case (where $a_{\alpha\beta mn} = \tau c_{\alpha\beta mn}$) it is possible to examine the character of the motion represented by the general solution (2·431) in greater detail.

From (2·355) we have $\mathrm{D}_r = \mathrm{B}_r^2/4\mathrm{C}_r$.

But

$$\mathrm{B}_r = \tau\mathrm{C}_r \quad \text{so that} \quad \mathrm{D}_r = \tfrac{1}{4}\tau\mathrm{B}_r. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2{\cdot}451)$$

Now D_r are arranged in increasing order of magnitude by definition; so that B_r also form a sequence of increasing magnitude.

The roots $p_r^{(1)}$, $p_r^{(2)}$ of the frequency equation are given by (2·354), which becomes in this case

$$p_r^{(1)},\, p_r^{(2)} = \tfrac{1}{2}\mathrm{B}_r\left\{-1 \pm \left[1 - \frac{4}{\tau\mathrm{B}_r}\right]^{\frac{1}{2}}\right\}. \quad . \ . \ . \ . \ . \ . \ . \quad (2{\cdot}452)$$

Thus the damping factor of the oscillatory terms is $e^{-\frac{1}{2}\mathrm{B}_r t}$; and the least heavily damped pair of oscillatory terms is the first pair, whose damping factor is $e^{-\frac{1}{2}\mathrm{B}_1 t}$.

When $\tfrac{1}{4}\tau\mathrm{B}_r = \mathrm{D}_r > 1$, the roots $p_r^{(1)}$, $p_r^{(2)}$ are real and the corresponding terms in the general solution represent aperiodic motions.

When D_r is just greater than one

$$p_r^{(1)},\, p_r^{(2)} = -\tfrac{1}{2}\mathrm{B}_r = -2/\tau \text{ approximately}, \quad \text{since } \tfrac{1}{4}\tau\mathrm{B}_r = 1 \text{ approximately.}$$

When $\mathrm{D}_r \gg 1$,

$$p_r^{(1)} = -1/\tau, \quad p_r^{(2)} = -\mathrm{B}_r \text{ approximately.}$$

Thus the least heavily damped of the aperiodic terms are the late $p_r^{(1)}$ terms, which have the approximately constant damping factor* $e^{-t/\tau}$.

Now the least powerful damping factor of the oscillatory terms is $e^{-\frac{1}{2}B_1 t}$.

Since $D_1 = \frac{1}{4}\tau B_1$, it follows that if

$$D_1 > \tfrac{1}{2}, \quad \text{then } \tfrac{1}{2}B_1 > 1/\tau\,;$$

and if

$$D_1 < \tfrac{1}{2}, \quad \text{then } \tfrac{1}{2}B_1 < 1/\tau.$$

We have already seen that the character of the motion in the free vibration depends on whether the critical number of the solid is greater or less than one (see Theorem III, p. 382) in the general case when no relation holds between the elastic and viscous coefficients. In the special case when $a_{\alpha\beta mn} = \tau c_{\alpha\beta mn}$, we see that there is an additional criterion as to whether the critical number of the solid is greater or less than $\frac{1}{2}$. We have the following theorem.

THEOREM V.—*If in a visco-elastic solid the elastic and viscous coefficients are connected by a relation* $a_{\alpha\beta mn} = \tau c_{\alpha\beta mn}$, *then the motion in the free vibration of the solid which has the least powerful damping factor* (*and appears to survive the longest*) *is damped oscillatory or aperiodic* (*elastic afterworking*) *according as the critical number is less than or greater than* $\frac{1}{2}$.

The results obtained in this section hold good for the torsional vibrations of thin isotropic visco-elastic cylinders, even when the relation $\lambda'/\lambda = \mu'/\mu$, which is the special form of the relation $a_{\alpha\beta mn} = \tau c_{\alpha\beta mn}$ for an isotropic solid, is not satisfied. For only the elastic coefficient μ (the rigidity) and the viscous coefficient μ' (coefficient of tangential viscosity) occur in the solution; and we have the relation $B_r = \tau_2 C_r$ (where $\tau_2 = \mu'/\mu$), from which the results of this section (2·4) follow.

2·5. *The problem of the "free" vibrations of a visco-elastic solid, a portion of whose surface is fixed.*

When a solid is allowed to vibrate with a portion of its surface fixed but the remainder of its surface free from traction, its vibrations are free in the sense that no work is done by externally applied forces during its vibration. Such vibrations have all the properties of true free vibrations which were deduced in the preceding sections (2·3) and (2·4).

The solution of the problem will, of course, be different; for though the differential equations (2·311) remain the same, the surface conditions (2·312) hold only over the free portion of the surface, being replaced by the surface conditions

$$\phi_\alpha = 0, \quad (\alpha = 1, 2, 3), \quad \ldots\ldots\ldots\ldots \quad (2.51)$$

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* The existence of this constant damping factor was noticed by O. E. MEYER in the case of the torsional vibrations of a thin cylinder—see MEYER, 'Pogg. Ann.' vol. 151, p. 108 (1874). Its existence in the approximate solution of the problem of the longitudinal vibrations of a thin cylinder was pointed out to me by Professor MILNE.

over the fixed portion of the surface. Now, all the results of the two preceding sections were obtained from the transformation of the volume integral (2·344). This transformation can be carried out in just the same way as before.

For the surface integral

$$\iint \nu_\beta \left(c_{\alpha\beta mn} + p_k\, a_{\alpha\beta mn}\right) e_{mn}{}^{(k)}\, \phi_\alpha{}^{(l)}\, ds$$

vanishes now, not because

$$\nu_\beta \left(c_{\alpha\beta mn} + p_k\, a_{\alpha\beta mn}\right) e_{mn}{}^{(k)} = 0$$

over the *whole* surface, but because this surface condition holds for the free portion of the surface and

$$\phi_\alpha{}^{(l)} = 0$$

over the fixed portion of the surface.

§3. THE VIBRATIONS OF THIN ISOTROPIC VISCO-ELASTIC CYLINDERS.

3·1. *Notation.*

The solution of problems concerning the vibration of a visco-elastic solid of particular shape is found to depend on some unsymmetrical feature of the shape. It is convenient therefore to drop the dummy suffixes and the summation convention of the tensor notation used in § 1 and § 2.

The directions of the axes are denoted by suffixes x, y, z. The displacement components u_α $(\alpha = 1, 2, 3)$ are denoted by u, v, w. The equations of motion (1·72) become

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}, \text{ etc.} \quad \ldots\ldots\ldots \quad (3{\cdot}11)$$

The stress-strain relations (1·91) for an isotropic visco-elastic solid become

$$p_{xx} = \left(\lambda + \lambda' \frac{\partial}{\partial t}\right) \Delta + 2\mu e_{xx}, \text{ etc.};$$

and

$$p_{yz} = p_{zy} = 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) e_{yz}, \text{ etc.} \quad \ldots\ldots\ldots \quad (3{\cdot}12)$$

The formula (1·22) for the strain components becomes

$$e_{xe} = \frac{\partial u}{\partial x}, \text{ etc.}; \qquad e_{yz} = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \text{ etc.} \quad \ldots\ldots\ldots \quad (3{\cdot}13)$$

In the above equations, ρ, λ, μ, λ', μ' are constants.

We remark that a problem concerning a visco-elastic solid differs only from the same problem concerning a perfectly elastic solid in that the LAMÉ elastic constants λ, μ are replaced by the operators

$$\lambda + \lambda' \frac{\partial}{\partial t}, \qquad \mu + \mu' \frac{\partial}{\partial t}.$$

It is often more convenient to introduce the viscous coefficients by means of the constants τ_1 and τ_2, defined by

$$\tau_1 = \frac{\lambda'}{\lambda}, \qquad \tau_2 = \frac{\mu'}{\mu}. \quad \ldots \ldots \ldots \quad (3\cdot14)$$

We call τ_1 and τ_2 *the visco-elastic constants of an isotropic solid.* They are of the dimensions of a time, and their significance will appear later.

3·2. *The free torsional vibrations of an isotropic visco-elastic circular cylinder.*

There is a solution of the equations of motion in which the torsional displacement at time t is given by

$$u = -y\theta(z, t), \qquad v = x\theta(z, t), \qquad w = 0,$$

the axis of z being taken as the axis of the rod.

Hence

$$p_{xx} = p_{yy} = p_{zz} = p_{xy} = 0\,;$$

and

$$p_{yz} = x\left(\mu + \mu'\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial z}\;; \qquad p_{zx} = -y\left(\mu + \mu'\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial z}.$$

The third of the equations of motion (3·11) is thus satisfied identically, and the first and second equations of motion become

$$\left(\mu + \mu'\frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial z^2} = \rho\frac{\partial^2\theta}{\partial t^2},$$

or

$$c_2^2\left(1 + \tau_2\frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial z^2} = \frac{\partial^2\theta}{\partial t^2}, \quad \ldots \ldots \ldots \quad (3\cdot21)$$

where $\tau_2 = \mu'/\mu$ and $c_2^2 = \mu/\rho$.

The components of traction across the curved surface $x^2 + y^2 = a^2$, vanish. For

$$\mathrm{P}_x = p_{\nu x} = \nu_x p_{xx} + \nu_y p_{yx} + \nu_z p_{zx} = 0, \quad \text{since } \nu_z = p_{xx} = p_{yx} = 0\,;$$

$$\mathrm{P}_y = p_{\nu y} = \nu_x p_{xy} + \nu_y p_{yy} + \nu_z p_{zy} = 0, \quad \text{since } \nu_z = p_{xy} = p_{yy} = 0\,;$$

$$\mathrm{P}_z = p_{\nu z} = \nu_x p_{xz} + \nu_y p_{yz} + \nu_z p_{zz}$$

$$= (x\nu_y - y\nu_x)\left(\mu + \mu'\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial z} = 0, \quad \text{since} \quad \frac{\nu_x}{\nu_y} = \frac{x}{y}.$$

Now $\theta(z, t) = \mathrm{A}e^{i\gamma z}\, e^{pt}$ is a solution of equation (3·21) if p is a root of

$$p^2 + c_2^2\gamma^2(1 + \tau_2 p) = 0. \quad \ldots \ldots \ldots \quad (3\cdot22)$$

Since γ occurs only as γ^2 in (3·22),

$$\theta(z, t) = \mathrm{B}\, e^{-i\gamma z}\, e^{pt}$$

is also a solution. Thus, changing the constants, we find that

$$\theta\,(z,t) = (\mathrm{A}\cos\gamma z + \mathrm{B}\sin\gamma z)\,(\mathrm{L}e^{p^{(1)}t} + \mathrm{M}e^{p^{(2)}t}) \quad . \; . \; . \; . \; . \quad (3\cdot23)$$

is a solution, where $p^{(1)}$, $p^{(2)}$ are roots of (3·22).

We consider two cases: (*a*) the cylinder free at both ends, $z = 0$ and $z = l$; (*b*) the cylinder fixed at $z = 0$ and free at $z = l$.

(*a*) It is found that the x and y components of traction vanish over the ends $z = 0$ and $z = l$ in the same way that they vanished over the curved surface. The condition that the z component of traction (equal to p_{zz}) should vanish at $z = 0$ and $z = l$ is

$$\mathrm{B} = 0, \quad \text{and} \quad \sin\gamma l = 0,$$

i.e.,

$$\mathrm{B} = 0, \quad \text{and} \quad \gamma = \frac{r\pi}{l}\ (r \text{ an integer}). \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3\cdot241)$$

Thus the solution (3·23) becomes

$$\theta\,(z,t) = \cos\frac{r\pi z}{l}\,(\mathrm{L}_r\,e^{p_r^{(1)}t} + \mathrm{M}_r\,e^{p_r^{(2)}t}), \quad . \; . \; . \; . \; . \; . \; . \quad (3\cdot242)$$

where $p_r^{(1)}$, $p_r^{(2)}$ are roots of

$$p^2 + \frac{c_2{}^2 r^2 \pi^2}{l^2}\,(1 + \tau_2 p) = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3\cdot243)$$

This equation for $p_r^{(1)}$, $p_r^{(2)}$ is the special case of the equation (2·348) which occurred in the general theory, namely,

$$\mathrm{A}_r p^2 + \mathrm{B}_r p + \mathrm{C}_r = 0, \text{ in which } \mathrm{A}_r = 1.$$

The critical number for the torsional vibrations of a circular cylinder free at both ends is the smallest of the

$$\mathrm{D}_r = \frac{\mathrm{B}_r{}^2}{4\mathrm{C}_r} = \frac{c_2{}^2 r^2 \pi^2 \tau_2{}^2}{4l^2},$$

i.e., $\dfrac{c_2{}^2 \pi^2 \tau_2{}^2}{4l^2}$, the value when $r = 1$.

Thus the critical number for the free torsional vibrations of a rod of length l is greater or less than one according as the length l is less than or greater than $\frac{1}{2}\pi\tau_2 c_2 = l_0$, say.

If $l < l_0$, the equation (3·243) can not have complex roots for any value of the integer r, and no oscillatory motion is possible.

If $l > l_0$, the equation (3·243) will have complex roots for a *finite* number of values of the integer. Free torsional vibrations will be the superposition of damped harmonic oscillations and aperiodic motions.

The length l_0 is the *critical length* of the cylinder for free torsional vibration.

By building up a general solution from (3·242) to obtain

$$\theta\,(z, t) = \sum_{r=1}^{\infty} \cos\frac{r\pi z}{l}\,(\mathrm{L}_r\, e^{p_r^{(1)}t} + \mathrm{M}_r\, e^{p_r^{(2)}t}), \quad . \; . \; . \; . \; . \; . \; . \quad (3\cdot 244)$$

it is possible to satisfy given initial conditions, for the torsion and rate of torsion, of the form

$$\theta\,(z, t) = \phi\,(z), \quad \text{for} \quad 0 < z < l \quad \text{when} \quad t = 0,$$

$$\frac{\partial}{\partial t}\,\theta\,(z, t) = \psi\,(z), \quad \text{for} \quad 0 < z < l \quad \text{when} \quad t = 0.$$

By suitable definition outside the interval $0 < z < l$, the functions $\phi\,(z)$ and $\psi\,(z)$ can be expanded in Fourier Series of the form

$$\phi\,(z) = \sum_r \alpha_r \cos\frac{r\pi z}{l}, \qquad \psi\,(z) = \sum_r \beta_r \cos\frac{r\pi z}{l};$$

so that the initial conditions are satisfied if

$$\mathrm{L}_r + \mathrm{M}_r = \alpha_r$$

and

$$p_r^{(1)}\,\mathrm{L}_r + p_r^{(2)}\,\mathrm{M}_r = \beta_r. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3\cdot 245)$$

The set of functions $\phi_\alpha^{(r)}$ $(\alpha = 1, 2, 3)$ which corresponded to the roots $p_r^{(1)}, p_r^{(2)}$ in § (2·4) are, in this example

$$-\,y\cos\frac{r\pi z}{l}, \quad x\cos\frac{r\pi z}{l}, \quad 0.$$

The orthogonal relation (2·444) is, in this example, the well-known orthogonal relation between the terms of a Fourier Series.

When $\mathrm{D}_r > 1$, the roots $p_r^{(1)}$, $p_r^{(2)}$ of equation (3·243) are

$$p_r^{(1)}, p_r^{(2)} = -\,\tfrac{1}{2}\tau_2\,\frac{c_2^{\,2}r^2\pi^2}{l^2} \pm \tfrac{1}{2}\,\frac{c_2 r\pi}{l}\left\{\frac{\tau_2^{\,2}c_2^{\,2}r^2\pi^2}{l^2} - 4\right\}^{\frac{1}{2}}$$

and become approximately

$$p_r^{(1)} = -\,\frac{1}{\tau_2}, \quad p_r^{(2)} = -\,\tau_2 c_2^{\,2}r^2\pi^2/l^2, \quad \text{when} \quad \mathrm{D}_r \gg 1.$$

When $\mathrm{D}_r < 1$, $p_r^{(1)}$, $p_r^{(2)}$ are given by

$$p_r^{(1)}, p_r^{(2)} = -\,\tfrac{1}{2}\tau_2\,\frac{c_2^{\,2}r^2\pi^2}{l^2} \pm i\,\frac{c_2 r\pi}{l}\left\{1 - \frac{\tau_2^{\,2}c_2^{\,2}r^2\pi^2}{4l^2}\right\}^{\frac{1}{2}}$$

and become approximately

$$p_r^{(1)}, p_r^{(2)} = -\,\tfrac{1}{2}\tau_2 c_2^{\,2}r^2\pi^2/l^2 \pm i\,\frac{c_2 r\pi}{l} \quad \text{when} \quad \mathrm{D}_r \ll 1.$$

The least powerful damping factor of the oscillatory terms is thus the damping factor of the first term, namely, $e^{-\frac{1}{2}\tau_2 \frac{c_2^2\pi^2}{l^2} \cdot t}$. The least powerful damping factor of the aperiodic terms is the approximately constant damping factor e^{-t/τ_2} of the late $p_r^{(1)}$ terms. Thus a damped oscillatory motion or an aperiodic motion (elastic afterworking) appears to survive the longest according as

$$\tfrac{1}{2}\tau_2 \frac{c_2^2\pi^2}{l^2} < \text{or} > \frac{1}{\tau_2};$$

i.e., as

$$D_1 = \frac{c_2^2\pi^2\tau_2^2}{4l^2} < \text{or} > \tfrac{1}{2};$$

or the criterion can be written as

$$l > \text{or} < \sqrt{2}l_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3\cdot246)$$

where l_0 is the critical length.

When $l \gg l_0$, so that the critical number $D_1 \ll 1$, the damped oscillatory motion represented by the first pair of terms appears to survive the longest, and after some time—during which the more heavily damped terms die down, and the observed damping is large—the angle of twist of the cylinder is represented approximately by

$$\theta\,(z, t) = \cos\frac{\pi z}{l} \,.\, e^{-\frac{1}{2}\tau_2\frac{c_2^2\pi^2}{l^2}t}\left\{\alpha_1 \cos\frac{c_2\pi}{l}\,t + \left[\frac{\beta_1 l}{c_2\pi} + \tfrac{1}{2}\,\frac{\tau_2 c_2\pi}{l}\,\alpha_1\right]\sin\frac{c_2\pi}{l}\,t\right\}. \quad . \quad . \qquad (3\cdot247)$$

To this order of approximation the viscosity of the cylinder does not affect the period of vibration

$$T = \frac{2\pi}{c_2\pi/l} = \frac{2l}{c_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3\cdot248)$$

but retards the phase by a time

$$\frac{\tau_2}{2\left\{1 + \left(\dfrac{\beta_1 l}{\alpha_1 c_2\pi}\right)^2\right\}} \text{ seconds.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3\cdot249)$$

(*b*) The cylinder is fixed at $z = 0$ and free at $z = l$.

The solution (3·23) for $\theta\,(z, t)$ satisfies the end conditions if $A = 0$ and $\cos\gamma l = 0$ *i.e.*, if

$$A = 0 \text{ and } \gamma = \frac{2r-1}{2l}\,\pi \quad (r \text{ an integer}) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3\cdot251)$$

Thus the solution (3·23) becomes

$$\theta\,(z, t) = \sin\frac{2r-1}{2l}\,\pi z\,(L_r\,e^{p_r^{(1)}t} + M_r\,e^{p_r^{(2)}t}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3\cdot252)$$

where p_r^1, p_r^2 are roots of

$$p^2 + \frac{c_2^2\,(2r-1)^2\pi^2}{4l^2}\,(1 + \tau_2 p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3\cdot253)$$

Equation (3·253) is the particular form of the equation $A_r p^2 + B_r p + C_r = 0$, occurring in the general theory.

The critical number for the "free" torsional vibrations of a circular cylinder, fixed at one end, is the smallest of the

$$D_r = \frac{\tau_2^2 c_2^2 (2r - 1)^2 \pi^2}{16 l^2},$$

i.e., $\dfrac{\tau_2^2 c_2^2 \pi^2}{16 l^2}$, the value then $r = 1$.

Thus the critical number is less than or greater than one according as the length of the cylinder is greater or less than $l_0' = \dfrac{\tau_2 c_2 \pi}{4}$. The length l_0' is the critical length of a circular cylinder for its "free" vibration when it is fixed at one end. This critical length is one half the critical length for the free torsional vibration with both ends free.

Results follow now in exactly the same way as in the preceding example when both ends were free. The constants L_r, M_r are again determined by the satisfying of the initial conditions. It is necessary, in this case, to expand the functions $\phi(z)$ and $\psi(z)$, which define the initial torsion and rate of torsion, in Fourier series of the form $\sum_r \alpha_r \sin \dfrac{(2r-1)\pi z}{2l}$; which is possible by suitable definition outside the interval $0 < z < l$.

3·3. *The free torsional vibrations of a thin isotropic visco-elastic non-circular cylinder.*

The solution of the problem for a circular cylinder breaks down in the case of a non-circular cylinder because the z-component of traction across the curved surface does not vanish. Now a visco-elastic solid behaves as a perfectly elastic solid when in equilibrium. A solution of the equilibrium problem, due to SAINT-VENANT,* shows that there is an axial displacement given by

$$w = \frac{\partial \theta}{\partial z} \phi(x, y),$$

where $\dfrac{\partial \theta}{\partial z}$ is the twist and is constant; and the function $\phi(x, y)$ is a plane harmonic function, determined by the fact that

$$\psi - \tfrac{1}{2}(x^2 + y^2) = \text{constant}$$

over the curved surface of the cylinder, where ψ is the function which is conjugate to ϕ.

The torsional couple is $C \dfrac{\partial \theta}{\partial z}$, where

$$C = \mu \iint \left(x^2 + y^2 + x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) dx\, dy$$

and is called the torsional rigidity.

---

* See LOVE, "Elasticity," Chap. 14.

To obtain a solution of the vibration problem, it is necessary to make a certain assumption—the same assumption that is made in the solution of the vibrational problem for a perfectly elastic cylinder.* The assumption is that the strain in a thin portion of the cylinder lying between two neighbouring cross sections is the same as if the thin portion were in equilibrium with the instantaneous twist. In the case of a perfectly elastic cylinder this assumption involves only neglecting the effects of the kinetic reactions on the strain distribution within the thin portion. In the case of a visco-elastic cylinder viscous stresses are called into play when the cylinder is vibrating; and it might be expected that we are also neglecting the effects of the viscous stresses on the strain distribution in the thin portion of the cylinder. But this is not the case for torsional vibrations, though it is for bending and longitudinal vibrations. For the statical problem of strain distribution in a twisted cylinder is purely geometrical, *e.g.*, the determination of $\phi(x, y)$. In effect, the viscous forces merely increase the value of the elastic stresses in the case of torsion, and do not affect the strain distribution in the thin portion of the cylinder that we are considering.

Making this assumption, we have the torsional couple equal to

$$\mathrm{C}\left(1+\tau_2\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial z}, \quad \text{where } \tau_2=\frac{\mu'}{\mu}, \qquad (3\cdot31)$$

as the elastic constant μ is replaced by the operator $\mu+\mu'\frac{\partial}{\partial t}$ when the cylinder is vibrating.

The equation of motion is formed by considering the motion of a thin cross-section of the cylinder lying between the z and $z+dz$ cross-sections in the standard configuration; and is

$$\rho\,\omega\,\mathrm{K}^2\frac{\partial^2\theta}{\partial t^2}=\mathrm{C}\left(1+\tau_2\frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial z^2}, \qquad (3\cdot32)$$

where ω is the cross-sectional area, and K is the radius of gyration of the cross-section about the centroid. The equation of motion can be written as

$$\frac{\partial^2\theta}{\partial t^2}=c_2'^2\left(1+\tau_2\frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial z^2}, \qquad (3\cdot33)$$

where

$$c_2'^2=\frac{\mathrm{C}}{\rho\,\omega\,\mathrm{K}^2}. \qquad (3\cdot34)$$

This is of the same form as the equation of motion (3·21) for a circular cylinder, c'_2 replacing c_2.

The solution of the problem of the torsional vibrations of a non-circular cylinder now follows in exactly the same way as for a circular cylinder.

---

* Love, "Elasticity" p. 427, where this assumption is discussed for the case of a perfectly elastic cylinder.

3·4. *The free longitudinal vibrations of a thin isotropic visco-elastic cylinder.*

When an elastic cylinder is in state of longitudinal tension the strain is such that

$$e_{xx} = e_{yy} = -\sigma e_{zz},$$

where σ, Poisson's ratio, is given by $\sigma = \dfrac{\lambda}{2(\lambda+\mu)}$.

Thus the determination of the strain distribution is not simply a geometrical problem as in the case of torsion but depends on the relative magnitudes of the Lamé elastic constants λ, μ. Quimby* has made the assumption that $e_{xx} = e_{yy} = -\sigma e_{zz}$ in his analysis of the problem of the longitudinal vibrations of thin visco-elastic cylinders. This is the equivalent assumption to the one that we made in § 3·3 for torsional vibrations. But in the case of longitudinal vibrations it is wrong because the viscous stresses seriously modify the strain distribution. For if the displacement components are proportional to e^{pt} say, then the effect of the viscous stresses on the analysis is to increase the values of λ, μ to $\lambda + \lambda' p$, $\mu + \mu' p$. Thus, unless $\lambda'/\lambda = \mu'/\mu$, the viscous stresses affect the strain distribution determined by Poisson's ratio, and Quimby's assumption leads to inaccurate results. Indeed, it is readily seen that, unless $\lambda'/\lambda = \mu'/\mu$, the traction over the curved surface will not vanish.

An approximate solution of the problem can, however, be obtained without difficulty. We look for a solution in which $e_{yz} = e_{zx} = e_{xy} = 0$; and $p_{xx} = p_{yy} = 0$. There will thus be no traction over the curved surface. We have

$$0 = p_{xx} = \left(\lambda + \lambda' \frac{\partial}{\partial t}\right) \Delta + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) e_{xx}$$

$$0 = p_{yy} = \left(\lambda + \lambda' \frac{\partial}{\partial t}\right) \Delta + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) e_{yy}$$

$$p_{zz} = \left(\lambda + \lambda' \frac{\partial}{\partial t}\right) \Delta + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) e_{zz}.$$

Eliminating e_{xx}, e_{yy} from these equations, we obtain

$$\left\{(\lambda+\mu) + (\lambda'+\mu')\frac{\partial}{\partial t}\right\} p_{zz} = \left(\mu + \mu' \frac{\partial}{\partial t}\right)\left\{(3\lambda+2\mu) + (3\lambda'+2\mu')\frac{\partial}{\partial t}\right\} e_{zz}.$$

The equation of motion

$$\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2}$$

becomes

$$\left(\mu + \mu' \frac{\partial}{\partial t}\right)\left\{(3\lambda+2\mu) + (3\lambda'+2\mu')\frac{\partial}{\partial t}\right\}\frac{\partial^2 w}{\partial z^2} = \rho\left\{(\lambda+\mu) + (\lambda'+\mu')\frac{\partial}{\partial t}\right\}\frac{\partial^2 w}{\partial t^2}. \quad (3{\cdot}41)$$

---

* 'Phys. Rev.,' vol. 25, p. 559 (1925). See introduction, p. 348.

The solution is only approximate as the other two equations of motion are not satisfied unless the cylinder is vanishingly thin—as is the case in the well-known solution of the same problem in the case of a perfectly elastic cylinder. The solution is most accurate for the graver modes of vibration, as the other two equations of motion are most nearly satisfied for the graver modes.

There is a solution of the equation of motion (3·41) of the form $w = \mathrm{A}e^{i\gamma z}e^{pt}$, if*

$$\rho p^2 + \gamma^2 \frac{(\mu + \mu' p)\{(3\lambda + 2\mu) + (3\lambda' + 2\mu')\, p\}}{\{(\lambda + \mu) + (\lambda' + \mu')\, p\}} = 0. \quad . \quad . \quad . \quad . \qquad (3\cdot 42)$$

Now γ occurs only as γ^2 in this equation, so that $w = \beta e^{-i\gamma z}e^{pt}$ is also a solution. Thus

$$w = (\mathrm{A}\cos\gamma z + \mathrm{B}\sin\gamma z)\, e^{pt}$$

is a solution. Suppose that the cylinder is free at both ends. Then we must have

$$\mathrm{B} = 0 \quad \text{and} \quad \gamma = \frac{r\pi}{l},$$

in order that $p_{zz} = 0$ at $z = 0$ and $z = l$.

Now equation (3·42) is a cubic equation for p in terms of γ. Let the roots be $p_r^{(1)}, p_r^{(2)}, p_r^{(3)}$ when $\gamma = \dfrac{r\pi}{l}$.

Thus we can build up the general solution

$$w = \sum_r \sin\frac{r\pi z}{l}\,(\mathrm{L}_r\, e^{p_r^{(1)}t} + \mathrm{M}_r\, e^{p_r^{(2)}t} + \mathrm{N}_r\, e^{p_r^{(3)}t}).$$

Now it is not possible to determine all the constants L_r, M_r, N_r by satisfying the initial conditions of displacement and velocity, and the solution is indeterminate. This appears to be a contradiction to theorem I (p. 370), concerning the uniqueness of solution. It seems that this is due to the fact that the solution is only approximate. We proceed to show that one of the roots is always large, provided that μ/μ' and λ/λ' are large (experiments on the damping of the vibrations of a torsional pendulum suggest a value of about 10^4 sec^{-1}†), while the other two roots may be small if the value of γ is sufficiently small.

We write the relation (3·42) as

$$\mathrm{F}(p) \equiv p^2\,[(\lambda + \mu) + (\lambda' + \mu')\, p] + \frac{\gamma^2}{\rho}\,[(\mu + \mu'\, p)\{(3\lambda + 2\mu) + (3\lambda' + 2\mu')\, p\}] = 0.$$

---

* The same $p - \gamma$ relation is obtained when the problem is investigated for a circular cylinder on the same lines as on p. 289 in LOVE's "Elasticity" for a perfectly elastic circular cylinder. This solution also is only approximate as there is a surface traction over the ends; and the approximation is again best for the graver modes. This comparison is a partial justification of the solution given above.

† *Cf.*, for example, HONDA and KONNO, 'Phil. Mag.,' vol. 42, p. 115 (1921). See also introduction.

Putting $p = -\frac{\mu}{\mu'}$, we have

$$F\left(-\frac{\mu}{\mu'}\right) = \left(\frac{\mu}{\mu'}\right)^2 \frac{\lambda\mu' - \lambda'\mu}{\mu'}.$$

Putting $p = -\frac{3\lambda + 2\mu}{3\lambda' + 2\mu'}$, we have

$$F\left(-\frac{3\lambda + 2\mu}{3\lambda' + 2\mu'}\right) = \left(\frac{3\lambda + 2\mu}{3\lambda' + 2\mu'}\right)^2 \frac{\lambda'\mu - \lambda\mu'}{3\lambda' + 2\mu'}.$$

Now $\mu' > 0$ and $3\lambda' + 2\mu' > 0$ by (1·93), so that $F\left(-\frac{\mu}{\mu'}\right)$ and $F\left(-\frac{3\lambda + 2\mu}{3\lambda' + 2\mu'}\right)$ have opposite signs.

Hence there must be a root lying between $-\frac{\mu}{\mu'}$ and $-\frac{3\lambda + 2\mu}{3\lambda' + 2\mu'}$, which is real and negative since, by (1·92), $\mu > 0$ and $3\lambda + 2\mu > 0$.

In view of the fact that the solution gives the best approximation for the graver modes of vibration, it seems that the third root lying between $-\frac{\mu}{\mu'}$ and $-\frac{3\lambda + 2\mu}{3\lambda' + 2\mu'}$ is an extraneous solution.

When the roots $p^{(1)}$, $p^{(2)}$ are small they are approximately the roots of the quadratic

$$\rho p^2 + \gamma^2 E (1 + \tau p) = 0, \quad \ldots \ldots \ldots \ldots \quad (3·43)$$

where E is YOUNG's modulus, and

$$\tau = \frac{\lambda\mu\tau_1 + (3\lambda^2 + 4\lambda\mu + 2\mu^2)\tau_2}{(\lambda + \mu)(3\lambda + 2\mu)}, \quad \text{where} \quad \tau_1 = \frac{\lambda'}{\lambda}, \quad \tau_2 = \frac{\mu'}{\mu}.$$

We call τ the "normal visco-elastic constant."

If the motion is so slow that the viscous stresses are small compared with the elastic stresses, the equation of motion (3·41) becomes approximately

$$\frac{\partial^2 w}{\partial t^2} = c^2\left(1 + \tau\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial z^2}, \quad \text{where} \quad c^2 = E/\rho\,; \quad \ldots \ldots \quad (3·44)$$

and the tension p_{zz} is given approximately by*

$$p_{zz} = E\left(1 + \tau\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial z}. \quad \ldots \ldots \ldots \ldots \quad (3·45)$$

It is not possible to discuss phenomena concerning the critical number and elastic afterworking by means of this solution, even by keeping the more exact form of the

---

* The quantity $E' = E\tau$ has been called the "coefficient of normal viscosity" by HONDA and KONNO (*loc. cit.*).

equation of motion (3·41). For when there is no oscillatory motion, the viscous stresses must be so large as to outweigh the elastic stresses. Thus, p must be of the order of $-\frac{\mu}{\mu'}$ and $-\frac{\lambda}{\lambda'}$, and this is the same order of magnitude as that of the third root of the equation (3·42) which we have already discarded as an extraneous solution on account of its magnitude.

The equation of motion (3·21) for the torsional vibrations of a circular visco-elastic cylinder, (3·33) for a non-circular cylinder, and the approximate equation of motion (3·44) for the slow longitudinal vibrations of thin rods, are all of the same form as the equation for the propagation of sound waves in a viscous medium, obtained by Stokes.*

For *slow* longitudinal vibrations of thin cylinders we obtain results from the approximate equation (3·44) which are similar to the results obtained in § 3·2 for the free torsional vibrations of a circular cylinder.

The forced torsional motion of a visco-elastic cylinder.

3·5. *The torsion of an isotropic visco-elastic cylinder at constant angular velocity.*

For simplicity the cylinder is supposed to be circular, though from the results obtained in § 3·3, it is clear that the analysis would hold equally well for a non-circular cylinder.

The equation of motion is (3·21), namely,

$$\frac{\partial^2 \theta}{\partial t^2} = c_2^2 \left(1 + \tau_2 \frac{\partial}{\partial t}\right) \frac{\partial^2 \theta}{\partial z^2}, \quad \ldots \ldots \ldots \ldots \quad (3\cdot51)$$

where $\theta(z, t)$ is the angle through which the section z is turned at time t.

The fixed end condition of the end $z = 0$ is

$$\theta(z, t) = 0, \quad \text{at } z = 0 \text{ for all } t. \quad \ldots \ldots \ldots \quad (3\cdot521)$$

The lower end ($z = l$) is constrained to move with constant angular velocity; and initially the cylinder is supposed to be in equilibrium in an unstrained position.

Thus

$$\theta(z, t) = \omega t, \quad \text{at } z = l. \quad \ldots \ldots \ldots \ldots \quad (3\cdot522)$$

The initial conditions are

$$\theta(z, t) = 0, \quad \text{when} \quad t = 0 \quad \text{and} \quad 0 \leq z \leq l, \quad \ldots \ldots \quad (3\cdot523)$$

and

$$\frac{\partial}{\partial t} \theta(z, t) = 0, \quad \text{when} \quad t = 0 \quad \text{and} \quad 0 \leq z < l. \quad \ldots \ldots \quad (3\cdot524)$$

---

* See Rayleigh, "Theory of Sound," vol. 2, p. 283.

There is a solution of (3·51),

$$\theta(z, t) = a_0 + a_1 z + a_2 t + a_3 zt,$$

but $a_0 = a_1 = a_2 = 0$ in order to satisfy (3·521) and (3·523).

The solution

$$\theta(z, t) = \sin \gamma_r z \, [e^{p_r^{(1)} t} - e^{p_r^{(2)} t}],$$

where $p_r^{(1)}$, $p_r^{(2)}$ are roots of

$$p^2 + c_2^2 \gamma_r^2 (1 + \tau_2 p) = 0,$$

also satisfies (3·521) and (3·523).

We build up the more general solution

$$\theta(z, t) = a_3 zt + \sum_r \mathrm{C}_r \sin \gamma_r z \, (e^{p_r^{(1)} t} - e^{p_r^{(2)} t}),$$

which also satisfies (3·521) and (3·523). Conditions (3·522) and (3·524) remain to be satisfied.

Condition (3·522) is satisfied if

$$a_3 = \omega/l \quad \text{and} \quad \gamma_r = r\pi/l.$$

Thus the solution becomes

$$\theta(z, t) = \omega \frac{zt}{l} + \sum_r \mathrm{C}_r \sin \frac{r\pi z}{l} (e^{p_r^{(1)} t} - e^{p_r^{(2)} t});$$

and condition (3·524) is satisfied if

$$\frac{\omega}{l} z + \sum_r \mathrm{C}_r \sin \frac{r\pi z}{l} (p_r^{(1)} - p_r^{(2)}) = 0 \quad \text{for} \quad 0 \leq z < l.$$

But

$$\sum_r \frac{(-1)^{r-1}}{r} \sin \frac{r\pi z}{l} = \tfrac{1}{2} \frac{\pi z}{l} \quad \text{for} \quad 0 \leq z < l.$$

Thus

$$[p_r^{(1)} - p_r^{(2)}] \, \mathrm{C}_r = \frac{2\omega (-1)^r}{\pi r},$$

where $p_r^{(1)}$, $p_r^{(2)}$ are the roots of

$$p^2 + c_2^2 \frac{r^2 \pi^2}{l^2} (1 + \tau_2 p) = 0.$$

Thus the solution for $\theta(z, t)$ is

$$\theta(z, t) = \omega \frac{z}{l} t + \sum_r \frac{2\omega}{\pi} \frac{(-1)^r}{r} \sin \frac{r\pi z}{l} \frac{e^{p_r^{(1)} t} - e^{p_r^{(2)} t}}{p_r^{(1)} - p_r^{(2)}} \quad \text{. . . . . (3·53)}$$

Now the couple which it is necessary to apply at the end $z = l$ to produce this motion is

$$\mathrm{L} = \mathrm{C}\left(1 + \tau_2 \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial z} \quad \text{at} \quad z = l,$$

where C is the torsional rigidity ($\mathrm{C} = \pi a^4 \mu/2$ for a circular section).

Thus

$$L = C\left\{\left[\frac{\omega}{l}(t+\tau)\right] + \sum_r \frac{2\omega}{l}\left[\frac{e^{p_r^{(1)}t} - e^{p_r^{(2)}t}}{p_r^{(1)} - p_r^{(2)}} + \tau_2 \frac{p_r^{(1)} e^{p_r^{(1)}t} - p_r^{(2)} e^{p_r^{(2)}t}}{p_r^{(1)} - p_r^{(2)}}\right]\right\}, \quad (3\cdot54)$$

where

$$p_r^{(1)}, p_r^{(2)} = -\tfrac{1}{2}\tau_2 c_2^2 \frac{r^2\pi^2}{l^2} \pm \frac{c_2 r\pi}{l}\left\{\frac{\tau_2^2 c^2 r^2 \pi^2}{4l^2} - 1\right\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (3\cdot55)$$

The series of terms of damped harmonics and aperiodic damping factors is not of any particular interest, as it is noticed that no term can be large compared with the rest. On account of the fact that all the terms are damped it is continuous.

If $l < \tau_2 c_2 \pi/2$, there are no damped harmonics and the torsional couple expressed by this series dies away aperiodically. If $l > \tau_2 c_2 \pi/2$, the torsional couple represented by this series will change sign before it dies away.

As time passes the torsional couple will be more and more accurately represented by

$$L = C\frac{\omega}{l}(t+\tau_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\cdot56)$$

Now in the case of a perfectly elastic cylinder, the expression for L becomes (putting $\tau = 0$)

$$L = C\left\{\frac{\omega}{l}t + \sum_r \frac{2\omega}{c_2 r\pi}\sin\frac{c_2 r\pi}{l}t\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\cdot57)$$

The terms in this series are now simple harmonic terms and do not die away. Further, the sum of these terms is discontinuous. We have

$$\sum_r \frac{1}{r}\sin\frac{c_2 r\pi}{l}t = \tfrac{1}{2}\left[(2n-1)\pi - \frac{c_2\pi}{l}t\right]$$

when

$$2n\pi > \frac{c_2\pi t}{l} > 2(n-1)\pi\,; \quad i.e.,\ 2n\frac{l}{c_2} > t > 2(n-1)\frac{l}{c_2} \quad (n \text{ an integer}).$$

The series terms which occur in the solution for $\theta(z, t)$ at any point of the cylinder represent a torsional vibration of the cylinder due to the sudden start, when the motion of the end point is given by

$$\frac{\partial}{\partial t}\theta(z, t) = \omega \quad (z = l)$$

and every other point of the cylinder is stationary.

The effect of the viscosity of the cylinder on the torsional couple required to produce the motion is :—

(1) The value of the torsional couple varies continuously with the time. The effect of the viscosity is to smooth out the discontinuity which occurs if the rod is elastic.

(2) When the vibration of the cylinder due to the discontinuous start has died down, the torsional couple L is $C(\omega/l)\tau_2$ more than would be required to maintain the cylinder in equilibrium in the instantaneous position. The value of the torsional couple has to anticipate the torsional displacement by a time of τ_2 seconds.

In the figs. 2 and 3 the torsional couple is plotted against the torsional displacement of the lower end. The dotted line gives the torsional couple necessary to keep the cylinder twisted and in equilibrium at the instantaneous displacements.

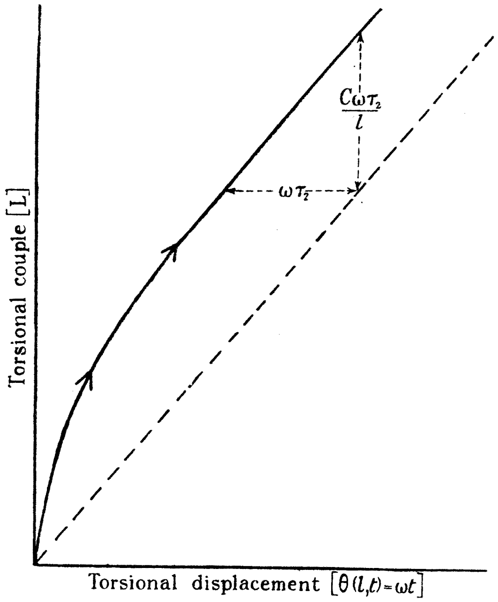

Fig. 1.—Visco-elastic cylinder.

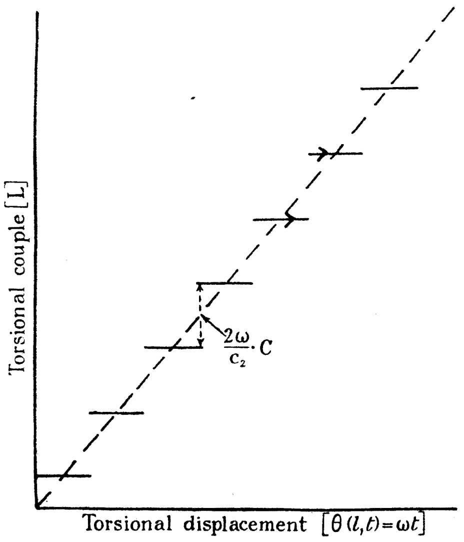

Fig. 2.—Perfectly elastic cylinder.

The graph for the visco-elastic cylinder has been drawn on the supposition that its length is less than $\frac{1}{2}\tau_2 c_2 \pi$. If the length of the cylinder is greater than $\frac{1}{2}\tau_2 c_2 \pi$, then the graph will start in a manner such as fig. 3.

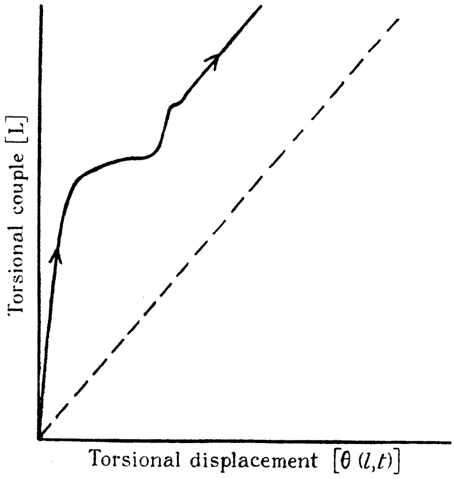

Fig. 3.

3·6. *The torsion of an isotropic visco-elastic cylinder by a torsional couple increasing linearly with the time* (L = Kt).

The equation of motion (3·51), the fixed end condition (3·521), and the condition of no initial displacement (3·523) are the same as in the last example in § 3·5.

The solution

$$\theta\,(z,t) = \sum_r \sin \gamma_r z\,(\mathrm{C}_r\, e^{p_r^{(1)}t} + \mathrm{D}_r\, e^{p_r^{(2)}t})$$

satisfies the fixed end condition.

The solution

$$\theta\,(z,t) = a_1 z + a_3 z t$$

satisfies the fixed end condition; and we build up the more general solution

$$\theta\,(z,t) = a_1 z + a_3 z t + \sum_r \sin \gamma_r z\,(\mathrm{C}_r\,.\, e^{p_r^{(1)}t} + \mathrm{D}_r\, e^{p_r^{(2)}t})$$

which also satisfies the fixed end condition.

The condition of the forced motion at the end $z = l$ is

$$\mathrm{K}t = \mathrm{L} = \mathrm{C}\left(1 + \tau_2 \frac{\partial}{\partial t}\right)\frac{\partial \theta}{\partial z} \text{ at } z = l; \qquad \ldots\ldots \quad (3\cdot61)$$

that is

$$\mathrm{K}t + \mathrm{C}\,\{(a_1 + a_3\tau_2) + a_3 t + \sum_r \gamma_r \cos \gamma_r l\,[(1 + \tau_2 p_r^{(1)})\,\mathrm{C}_r\, e^{p_r^{(1)}t} + (1 + \tau_2 p_r^{(2)})\,\mathrm{D}_r\, e^{p_r^{(2)}t}]\}.$$

This is satisfied if

$$a_3 = \mathrm{K}/\mathrm{C},\; a_1 = -\frac{\mathrm{K}\tau_2}{\mathrm{C}};\; \text{ and } \cos \gamma_r l = 0,\; i.e.,\; \gamma_r = \frac{(2r-1)\,\pi}{2l}.$$

The initial conditions

$$\theta\,(z,t) = 0, \text{ when } t = 0, \text{ for } 0 \leq z \leq l \qquad \ldots\ldots \quad (3\cdot621)$$

and

$$\frac{\partial}{\partial t}\,\theta\,(z,t) = 0, \text{ when } t = 0, \text{ for } 0 \leq z \leq l \qquad \ldots\ldots \quad (3\cdot622)$$

remain to be satisfied.

The initial conditions reduce to

$$-\frac{\mathrm{K}}{\mathrm{C}}\,\tau_2 z + \sum_r (\mathrm{C}_r + \mathrm{D}_r) \sin \frac{(2r-1)\,\pi z}{2l} = 0, \text{ for } 0 \leq z \leq l;$$

and

$$\frac{\mathrm{K}}{\mathrm{C}}\, z + \sum_r (p_r^{(1)}\,\mathrm{C}_r + p_r^{(2)}\,\mathrm{D}_r) \sin \frac{(2r-1)\pi z}{2l} = 0, \text{ for } 0 \leq z \leq l.$$

But

$$z = \frac{8l}{\pi^2} \sum_r \frac{(-1)^{r-1}}{(2r-1)^2} \sin \frac{(2r-1)\,\pi z}{2l} \text{ when } -2l < z < 2l.$$

Thus the initial conditions are satisfied if

$$\mathrm{C}_r + \mathrm{D}_r = \frac{\mathrm{K}}{\mathrm{C}}\,\tau_2\,\frac{8l}{\pi^2}\,\frac{(-1)^{r-1}}{(2r-1)^2}$$

and

$$p_r^{(1)}C_r + p_r^{(2)}D_r = -\frac{K}{C}\frac{8l}{\pi^2}\frac{(-1)^{r-1}}{(2r-1)^2},$$

where $p_r^{(1)}$, $p_r^{(2)}$ are the roots of

$$p^2 + \left[\frac{(2r-1)\pi}{2l}\right]^2 c_2^2(1+\tau_2 p) = 0.$$

The solution for $\theta(z, t)$ is thus

$$\theta(z, t) = \frac{K}{C}z(t-\tau_2) + \sum_r \sin\frac{(2r-1)\pi z}{2l}[C_r e^{p_r^{(1)}t} + D_r e^{p_r^{(2)}t}], \quad . \quad . \quad . \quad . \quad . \quad (3\cdot63)$$

where

$$\left.\begin{aligned} C_r &= -\frac{K}{C}\frac{8l}{\pi^2}\frac{(-1)^{r-1}}{(2r-1)^2}\frac{1+\tau_2 p_r^{(2)}}{p_r^{(1)}-p_r^{(2)}}, \\ \text{and} \qquad D_r &= +\frac{K}{C}\frac{8l}{\pi^2}\frac{(-1)^{r-1}}{(2r-1)^2}\frac{1+\tau_2 p_r^{(1)}}{p_r^{(1)}-p_r^{(2)}}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\cdot64)$$

Again the series terms are not of particular interest, as the motion they represent is damped, and it is easily seen to be continuous on account of the exponential damping factors which occur with each term. As time passes the motion of the end $z = l$ is more and more accurately represented by

$$\theta(l, t) = \frac{K}{C}l(t-\tau_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\cdot65)$$

If the cylinder is perfectly elastic, $\tau_2 = 0$, and we have

$$\theta(l, t) = \frac{K}{C}zt - \frac{16l^3}{\pi^3 c_2}\frac{K}{C}\sum_r \frac{1}{(2r-1)^3}\sin\frac{(2r-1)\pi c_2 t}{2l} \qquad (3\cdot66)$$

The terms in the series are not damped but the motion they represent is continuous. When $t < 2l/c$

$$\sum_r \frac{1}{(2r-1)^3}\sin\frac{(2r-1)\pi c_2 t}{2l} = \frac{\pi^3}{16l^3}\tfrac{1}{2}c_2^2 t^2.$$

The effect of the viscosity of the cylinder on the torsional displacement at the end $z = l$ due to the torsional couple, $L = Kt$, is :—

(1) While in the case of a perfectly elastic cylinder there will be a vibration about the steady motion owing to the manner of starting, in the case of a visco-elastic cylinder this vibration will subside, leaving only the steady motion.

(2) When the vibratory motion has subsided the steady motion is expressed by $\theta(l, t) = (K/C)\,l(t-\tau_2)$, which is the equilibrium torsional displacement due to a torsional couple $K(t-\tau_2)$. The torsional displacement thus lags behind the torsional couple by a time of τ_2 seconds.

3·7. *The hysteresis loop for the torsion of an isotropic visco-elastic cylinder.*

We can plot the torsional displacement of the lower end of the cylinder against the torsional couple applied to that end during a cycle of changes. The results of either § 3·5 or § 3·6 can be used. We use the results of § 3·6 for twisting by a torsional couple $L = Kt$.

The solution (3·63) holds just the same if K is negative. Further if the cylinder is initially twisted so that

$$\theta(z, t) = Az \quad \text{when} \quad t = 0,$$

the solution is obtained by adding Az to the value for $\theta(z, t)$ expressed by (3·63).

We take the cylinder through the following cycle of changes.

(1) The cylinder is initially twisted by a constant couple $L = KT$. A couple of magnitude $L = K(T - t)$ is then applied for a time $2T$ when the value of the couple has become $L = -KT$.

(2) The couple is maintained at this magnitude for some time. When the magnitude of the torsional couple attains this value the cylinder will have a torsional displacement given approximately by $\theta(l, t) = -\frac{Kl}{C}(T - \tau_2)$. When the couple is kept at this constant magnitude, since the torsional displacement has not the appropriate value for an equilibrium position torsional vibrations will be set up in the cylinder. These will die down until the torsional displacement is given by $\theta(l, t) = -\frac{Kl}{C}T$.

(3) The cylinder is now twisted by a couple of magnitude $L = K(t - T)$ for a time $2T$, when the magnitude of the couple becomes $L = KT$, the value of the couple at the start of the cycle of changes. The torsional displacement is now given approximately by $\theta(l, t) = \frac{Kl}{C}(T - \tau_2)$, which differs from the equilibrium displacement $\theta(l, t) = \frac{KlT}{C}$ of the cylinder under a couple $L = KT$. Thus torsional vibrations are set up, and they die down until the torsional displacement finally assumes its initial value, given by $\theta(l, t) = \frac{KlT}{C}$. This completes the cycle of changes.

In fig. 4 the torsional couple is plotted against the torsional displacement $\theta(l, t)$ during this cycle of changes. The dotted line is the graph of the torsional couple plotted against the corresponding equilibrium torsional displacement of the end $z = l$. The hysteresis loop has been drawn on the assumption that the length of the cylinder is less than $\frac{\tau_2 c_2 \pi}{4}$. If it is greater than $\frac{\tau_2 c_2 \pi}{4}$, then the ends of the loop will be of more complicated shape (*cf.*, for instance the graph, fig. 3).

The horizontal width of the loop is approximately equal to $2\frac{Kl\tau_2}{C}$. Thus the thickness of the loop is proportional to K, which is the rate of change of the couple L with the time. The hysteresis loop is thus thickest for the fastest cycles of changes. When the cycle of changes is made infinitely slowly, the thickness of the loop is infinitesimally small. For indefinitely fast cycles of changes the thickness of the loop is indefinitely great. We remember, however, that the theory of visco-elasticity only gives an accurate representation of the behaviour of the solid if the rate of straining is less than the "upper visco-elastic limit"; and if the effects of plasticity are not absent, only if the rate of straining is greater than the "lower visco-elastic limit" (see § 1.6). An experimental investigation of the hysteresis loop appears to be a ready method of determining the visco-elastic limits.

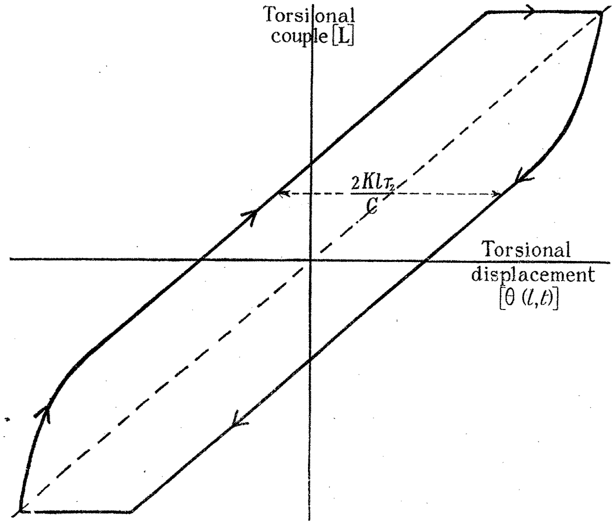

FIG. 4.—Hysteresis Loop.

Summary.

The object of the paper is to investigate the behaviour of "visco-elastic" solids, the relation between the stress and strain in their interior, and the character of their vibrations under given surface conditions.

An account is given of work on imperfectness of elasticity since 1834. The results of previous investigations are examined with a view to the formulation of a general theory of the behaviour of imperfectly elastic solids.

By the complete application of the principle of virtual work to a strained and straining imperfectly elastic solid, a general theory of imperfect elasticity—of the type called visco-elasticity—is developed. The modification of the stress-strain relations in a visco-elastic solid due to thermo-elastic causes is worked out.

Some general theorems are proved concerning the vibrations of visco-elastic solids. It is shown that any solution of the equations of motion, subject to given initial conditions of displacement and velocity and subject to the surface conditions which hold for forced or free vibrations, is a unique solution. A *critical number* for a visco-elastic solid is defined. It is shown that, if the critical number is greater than one, then the solid can not execute an oscillatory motion when there is no surface traction. When the critical number of the solid is less than one it is shown that the motion of the solid in its free vibration is the superposition of damped harmonic oscillations and aperiodic motions. For solids of simple shape, it is shown that there is a *critical size* such that solids of smaller size cannot execute oscillatory motion when free from surface traction.

The general theorems are illustrated by some examples on the torsional and longitudinal vibrations of thin isotropic cylinders. Two problems of *forced* torsional motion are analysed, and the results compared with those for perfectly elastic cylinders. The hysteresis loop for a closed cycle of changes is constructed from these results.